Hierarchical Bayesian modelling of volatile environments

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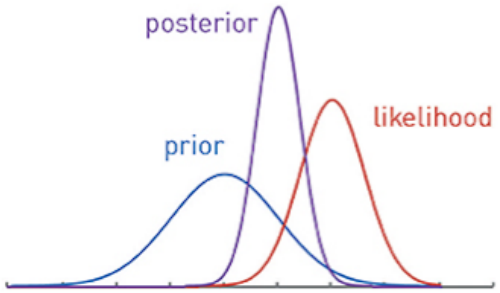
Outline

- The inferential (i.e., Bayesian) brain
- Reduction of Bayesian inference to precision-weighted prediction errors
- Non-stationary environments and the hierarchical Gaussian filter (HGF)
- Experimental studies and results

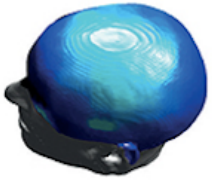
Computational modelling and the inferential brain

Bayes' Theorem

$$p(x|y,m) = \frac{p(y|x,m)p(x|m)}{p(y|m)}$$



Generative models as computational assays



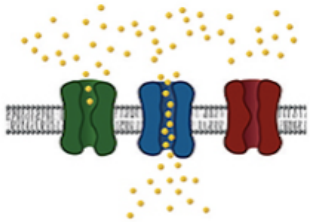
measurement y

forward model

$p(y|x,m)p(x|m)$

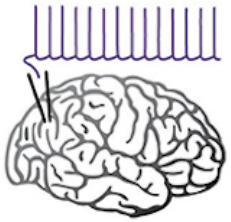
$p(x|y,m)$

inference



hidden system state x

Perception as the inversion of a generative model



neuronal activity y

forward model

$p(y|x,m)p(x|m)$

$p(x|y,m)$

perception

environmental state x

Stephan et al. (2016), Front. Hum. Neurosci., 10:550

“Every good regulator of a system must be a model of that system” (Conant & Ashby, 1970)

Abstract:

«*The design of a complex regulator often includes the making of a model of the system to be regulated. The making of such a model has hitherto been regarded as optional, as merely one of many possible ways.*

In this paper a theorem is presented which shows, under very broad conditions, that any regulator that is maximally both successful and simple must *be isomorphic with the system being regulated. (The exact assumptions are given.) Making a model is thus necessary.*

The theorem has the interesting corollary that ***the living brain****, so far as it is to be successful and efficient as a regulator for survival,* **must *proceed****, in learning,* ***by the formation of a model (or models) of its environment.***»

What the inferential brain does: Bayesian inference

- «Bayesian inference» simply means inference on uncertain quantities according to the rules of probability theory (i.e., according to logic).

- Agents who use Bayesian inference will make better predictions (provided they have a good model of their environment), which will give them an evolutionary advantage.

- We may therefore assume that evolved biological agents use Bayesian inference, or a close approximation to it.

- But how can we reduce Bayesian inference to a simple algorithm that can be implemented by neurons?

Before we return to Bayes: a very simple example of updating in response to new information

Imagine the following situation:

You're on a boat, you're lost in a storm and trying to get back to shore. A lighthouse has just appeared on the horizon, but you can only see it when you're at the peak of a wave. Your GPS etc., has all been washed overboard, but what you can still do to get an idea of your position is to measure the angle between north and the lighthouse. These are your measurements (in degrees):

76, 73, 75, 72, 77

What number are you going to base your calculation on?

Right. The mean: 74.6. How do you calculate that?

Updating the mean of a series of observations

The usual way to calculate the mean $\bar{x}$ of $x_1, x_2, \ldots, x_n$ is to take

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

This requires you to remember all x_i, which can become inefficient. Since the measurements arrive sequentially, we would like to update $\bar{x}$ sequentially as the x_i come in – without having to remember them.

It turns out that this is possible. After some algebra (see next slide), we get

$$\bar{x}_{n+1} = \bar{x}_n + \frac{1}{n+1}(x_{n+1} - \bar{x}_n)$$

Updating the mean of a series of observations

Proof of sequential update formula:

$$\bar{x}_{n+1} = \frac{1}{n+1}\sum_{i=1}^{n+1} x_i = \frac{x_{n+1}}{n+1} + \frac{1}{n+1}\sum_{i=1}^{n} x_i = \frac{x_{n+1}}{n+1} + \frac{n}{n+1}\underbrace{\frac{1}{n}\sum_{i=1}^{n} x_i}_{=\bar{x}_n} =$$

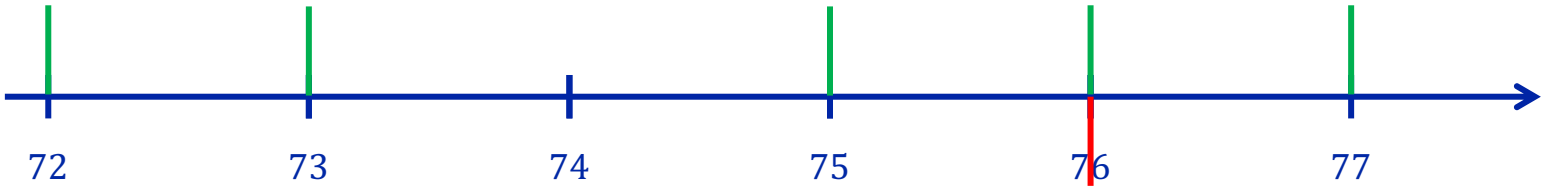
$$= \frac{x_{n+1}}{n+1} + \frac{n}{n+1}\bar{x}_n = \bar{x}_n + \frac{x_{n+1}}{n+1} + \frac{n}{n+1}\bar{x}_n - \frac{n+1}{n+1}\bar{x}_n =$$

$$= \bar{x}_n + \frac{1}{n+1}(x_{n+1} + (n - n - 1)\bar{x}_n) = \bar{x}_n + \frac{1}{n+1}(x_{n+1} - \bar{x}_n)$$

q.e.d.

Updating the mean of a series of observations

The seqential updates in our example now look like this:



$\bar{x}_1 = 76$

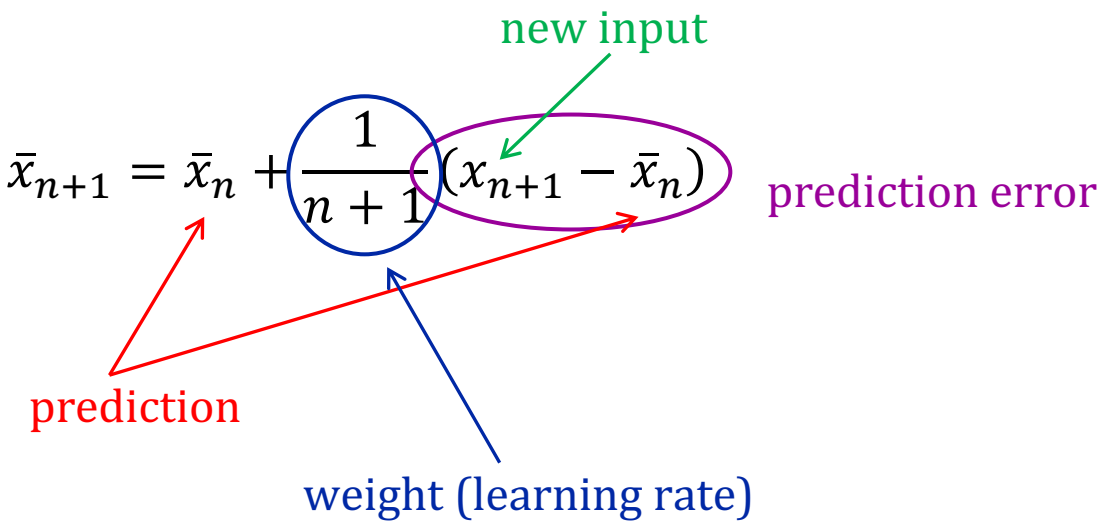
$\bar{x}_2 = 76 + \frac{1}{2}(73 - 76) = 74.5$

$\bar{x}_3 = 74.5 + \frac{1}{3}(75 - 74.5) = 74.\overline{6}$

$\bar{x}_4 = 74.\overline{6} + \frac{1}{4}(72 - 74.\overline{6}) = 74$

$\bar{x}_5 = 74 + \frac{1}{5}(77 - 74) = 74.6$

What are the building blocks of the updates we've just seen?



Is this a general pattern?

More specifically, does it generalize to Bayesian inference?

Indeed, it turns out that in many cases, Bayesian inference can be based on parameters that are updated using **precision-weighted prediction errors**.

Updates in a simple Gaussian model

Think boat, lighthouse, etc., again, but now we're doing Bayesian inference.

Before we make the next observation, our belief about the true value of the parameter ϑ can be described by a Gaussian prior:

$$p(\vartheta) \sim \mathcal{N}(\mu_\vartheta, \pi_\vartheta^{-1})$$

The likelihood of an observation x is also Gaussian, with precision π_ε :

$$p(x|\vartheta) \sim \mathcal{N}(\vartheta, \pi_\varepsilon^{-1})$$

Bayes' rule now tells us that the posterior is Gaussian again:

$$p(\vartheta|x) = \frac{p(x|\vartheta)p(\vartheta)}{\int p(x|\vartheta')p(\vartheta')\mathrm{d}\vartheta'} \sim \mathcal{N}\left(\mu_{\vartheta|x}, \pi_{\vartheta|x}^{-1}\right)$$

Updates in a simple Gaussian model

Here's how the updates to the sufficent statistics μ and π describing our belief look like:

$$\pi_{\vartheta|x} = \pi_{\vartheta} + \pi_{\varepsilon}$$

$$\mu_{\vartheta|x} = \mu_{\vartheta} + \frac{\pi_{\varepsilon}}{\pi_{\vartheta|x}}(x - \mu_{\vartheta})$$

prediction error

prediction

weight (learning rate)=$\frac{\text{how much we're learning here}}{\text{how much we already know}}$

The mean is updated by an uncertainty-weighted (more specifically: precision-weighted) prediction error.

The size of the update is proportional to the likelihood precision and inversely proportional to the posterior precision.

This pattern is not specific to the univariate Gaussian case, but generalizes to Bayesian updates for all exponential families of likelihood distributions with conjugate priors (i.e., to all formal descriptions of inference you are ever likely to need).

Reduction to mean updating

Reminder (Gaussian update):

$$\mu_{\vartheta|x} = \mu_\vartheta + \frac{\pi_\varepsilon}{\pi_{\vartheta|x}}(x - \mu_\vartheta) = \mu_\vartheta + \frac{\pi_\varepsilon}{\pi_\vartheta + \pi_\varepsilon}(x - \mu_\vartheta)$$

Reducing by $\pi_\varepsilon$the fraction of precisions that make the learning rate, we get

$$\mu_{\vartheta|x} = \mu_\vartheta + \frac{1}{\frac{\pi_\vartheta}{\pi_\varepsilon} + 1}(x - \mu_\vartheta)$$

As we shall see, this is the equation for updating an arithmetic mean, but with the number of observations n replaced by $\frac{\pi_\vartheta}{\pi_\varepsilon}$.

This shows that Bayesian inference on the mean of a Gaussian distribution entails nothing more than updating the arithmetic mean of observations with $\frac{\pi_\vartheta}{\pi_\varepsilon} =: \nu$ as a proxy for the number of prior observations, i.e. for the **weight of the prior relative to the observation**.

Generalization to all exponential families of distributions

Many of the most widely used probability distributions are families of exponential distributions.

For example, the Gaussian distribution is an exponential family of distributions (and so are the beta, gamma, binomial, Bernoulli, multinomial, categorical, Dirichlet, Wishart, Gaussian-gamma, log-Gaussian, multivariate Gaussian, Poisson, and exponential distributions, among others). This means it can be written the following way:

$$p(\boldsymbol{x}|\boldsymbol{\vartheta}) = h(\boldsymbol{x}) \exp(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \boldsymbol{T}(\boldsymbol{x}) - A(\boldsymbol{\vartheta})) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(x-\mu)^2}{2\sigma}\right)$$

with

$$\boldsymbol{x} = x, \qquad \boldsymbol{\vartheta} = (\mu, \sigma)^{\mathrm{T}}, \qquad h(\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}}, \qquad \boldsymbol{\eta}(\boldsymbol{\vartheta}) = \left(\frac{\mu}{\sigma}, -\frac{1}{2\sigma}\right)^{\mathrm{T}}, \qquad \boldsymbol{T}(\boldsymbol{x}) = (x, x^2)^{\mathrm{T}}, \qquad A(\boldsymbol{\vartheta}) = \frac{\mu^2}{\sigma} + \frac{\ln \sigma}{2}$$

This allows us to look at Bayesian belief updating in a very general way for all exponential families of distributions.

Generalization to all exponential families of distributions (Mathys, 2016)

Our likelihood is an exponential family in its general form:

$$p(\boldsymbol{x}|\boldsymbol{\vartheta}) = h(\boldsymbol{x}) \exp(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \boldsymbol{T}(\boldsymbol{x}) - A(\boldsymbol{\vartheta}))$$

The vector $\boldsymbol{T}(\boldsymbol{x})$ (a function of the observation $\boldsymbol{x}$) is called the sufficient statistic.

For the prior, we may assume that we have made ν observations with sufficient statistic $\boldsymbol{\xi}$:

$$p(\boldsymbol{\vartheta}|\boldsymbol{\xi}, \nu) = z(\boldsymbol{\xi}, \nu) \exp\big(\nu(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \boldsymbol{\xi} - A(\boldsymbol{\vartheta}))\big) \quad \text{(where } z(\xi, \nu) \text{ is a normlization constant)}$$

It then turns out that the posterior has the same form, but with an updated $\boldsymbol{\xi}$ and ν replaced with $\nu + 1$:

$$p(\boldsymbol{\vartheta}|\boldsymbol{x}, \boldsymbol{\xi}, \nu) = z(\boldsymbol{\xi}', \nu + 1) \exp\big((\nu + 1)(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \boldsymbol{\xi}' - A(\boldsymbol{\vartheta}))\big)$$

$$\boldsymbol{\xi}' = \boldsymbol{\xi} + \frac{1}{\nu + 1}(\boldsymbol{T}(\boldsymbol{x}) - \boldsymbol{\xi})$$

Proof of the update equation

$$\overbrace{p(\boldsymbol{\vartheta}|\boldsymbol{x},\boldsymbol{\xi},\nu)}^{\text{posterior}} \propto \overbrace{p(\boldsymbol{x}|\boldsymbol{\vartheta})}^{\text{likelihood}}\ \overbrace{p(\boldsymbol{\vartheta}|\boldsymbol{\xi},\nu)}^{\text{prior}}$$

$$= h(\boldsymbol{x}) \exp(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \boldsymbol{T}(\boldsymbol{x}) - A(\boldsymbol{\vartheta})) z(\boldsymbol{\xi},\nu) \exp\big(\nu(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \boldsymbol{\xi} - A(\boldsymbol{\vartheta}))\big)$$

$$\propto \exp\big(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot (\boldsymbol{T}(\boldsymbol{x}) + \nu\boldsymbol{\xi}) - (\nu+1)A(\boldsymbol{\vartheta})\big)$$

$$= \exp\left((\nu+1)\left(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \frac{1}{\nu+1}(\boldsymbol{T}(\boldsymbol{x}) + \nu\boldsymbol{\xi}) - A(\boldsymbol{\vartheta})\right)\right)$$

$$= \exp\left((\nu+1)\left(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \left(\boldsymbol{\xi} + \frac{1}{\nu+1}(\boldsymbol{T}(\boldsymbol{x}) + \nu\boldsymbol{\xi} - (\nu+1)\boldsymbol{\xi})\right) - A(\boldsymbol{\vartheta})\right)\right)$$

$$= \exp\left((\nu+1)\left(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \left(\underbrace{\boldsymbol{\xi} + \frac{1}{\nu+1}(\boldsymbol{T}(\boldsymbol{x}) - \boldsymbol{\xi})}_{=:\boldsymbol{\xi}'}\right) - A(\boldsymbol{\vartheta})\right)\right)$$

$$\implies \quad p(\boldsymbol{\vartheta}|\boldsymbol{x},\boldsymbol{\xi},\nu) = z(\boldsymbol{\xi}',\nu') \exp\Big(\nu'\big(\boldsymbol{\eta}(\boldsymbol{\vartheta}) \cdot \boldsymbol{\xi}' - A(\boldsymbol{\vartheta})\big)\Big)$$

$$\text{with } \nu' \coloneqq \nu + 1, \qquad \boldsymbol{\xi}' \coloneqq \boldsymbol{\xi} + \frac{1}{\nu+1}(\boldsymbol{T}(\boldsymbol{x}) - \boldsymbol{\xi})$$

q.e.d.

Some examples

Univariate Gaussian model with unkown mean but **known precision** (our example from the beginning):

$$\boldsymbol{T}(x) = x$$

This means updating beliefs about the mean simply requires tracking the mean of observations

Univariate Gaussian model with unkown mean and unkown precision:

$$\boldsymbol{T}(x) = (x, x^2)^{\mathrm{T}}$$

Updating beliefs about both mean and precision of a Gaussian requires tracking the means of observations and squared observations; this amounts to the first and second moments by which a Gaussian distribution is fully characterized.

In the **multivariate Gaussian** case we have $\boldsymbol{T}(\boldsymbol{x}) = \left(\boldsymbol{x}, \boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\right)^{\mathrm{T}}$

Some examples

Bernoulli model (one out of two possible outcomes, coded as 0 and 1; e.g., coin flipping):

$$\boldsymbol{T}(x) = x$$

The prior here turns out to be a **beta distribution** corresponding to ν pseudo-observations with mean ξ. All we need to do to get the posterior (i.e., to update our belief) is to update the mean as new observations come in.

Categorical model (one out of several possible outcomes, with the observed outcome coded as 1, the rest as 0)

$$\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{x}$$

The prior and posterior here are **Dirichlet distributions**, and again, all we need to do to update beliefs that have a Dirichlet form is to track the means of observed succeses (1) and failures (0).

Some examples

Beta model (an outcome bounded between 0 and 1):

$$\boldsymbol{T}(x) = (\ln x\,, \ln(1-x))^{\mathrm{T}}$$

Gamma model (an outcome bounded below at 0):

$$\boldsymbol{T}(x) = (\ln x\,, x)^{\mathrm{T}}$$

… so in sum:

… we have dealt (among others) with beliefs about states that are

- binary (Bernoulli)
- categorical
- bounded on both sides (beta)
- bounded on one side (gamma)
- and unbounded (Gaussian)

This includes most kinds of states we can have beliefs about. Notably,

- **All Bayesian (i.e., probabilistic, rational) updates of such beliefs take the form of precision-weighted prediction errors.**

- **These prediction errors and their precision weights are easy to compute.**

- **The prediction errors are simple functions of inputs.**

Limitations

- Examples of distributions that are not exponential families: Student's *t*, Cauchy

- These distributions are popular because of their «fat tails». However, fat tails can also be achieved with appropriate hierarchies of Gaussians (cf. the hierarchical Gaussian filter, HGF)

- A further kind of distributions that are not exponential families are found in mixture models.

- Such models are popular because of they provide multimodal distributions. But again, appropriate hierarchies of distributions may save the day.

How to reveal the precision-weighting of prediction errors when simple exponential-family likelihoods will not do

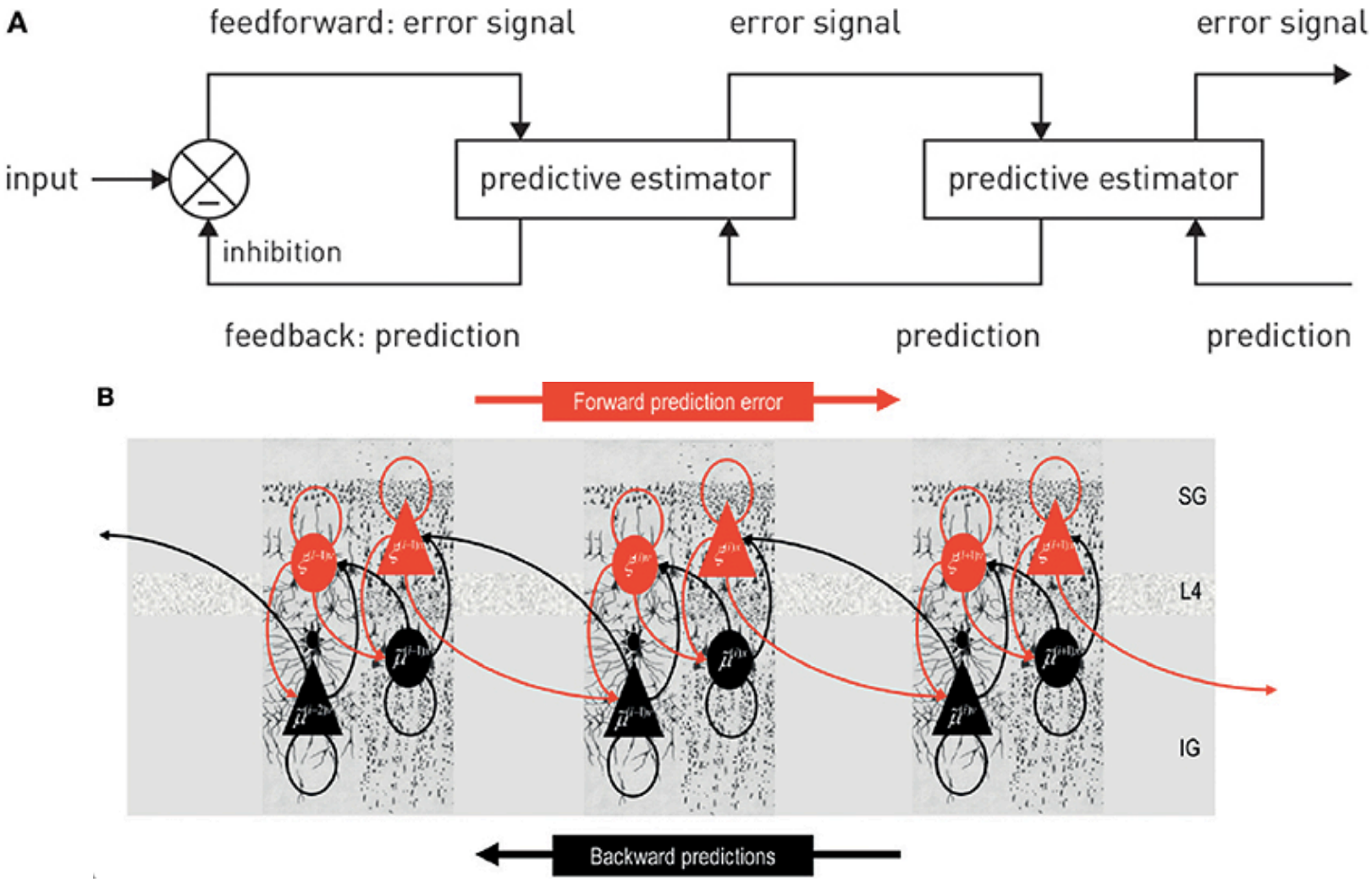
- Formulate the problem hierarchically (i.e., imitate evolution: when it built a brain that supports a mind which is a model of its environment, it came up with a (largely) hierarchical solution)

- Separate levels using a mean-field approximation

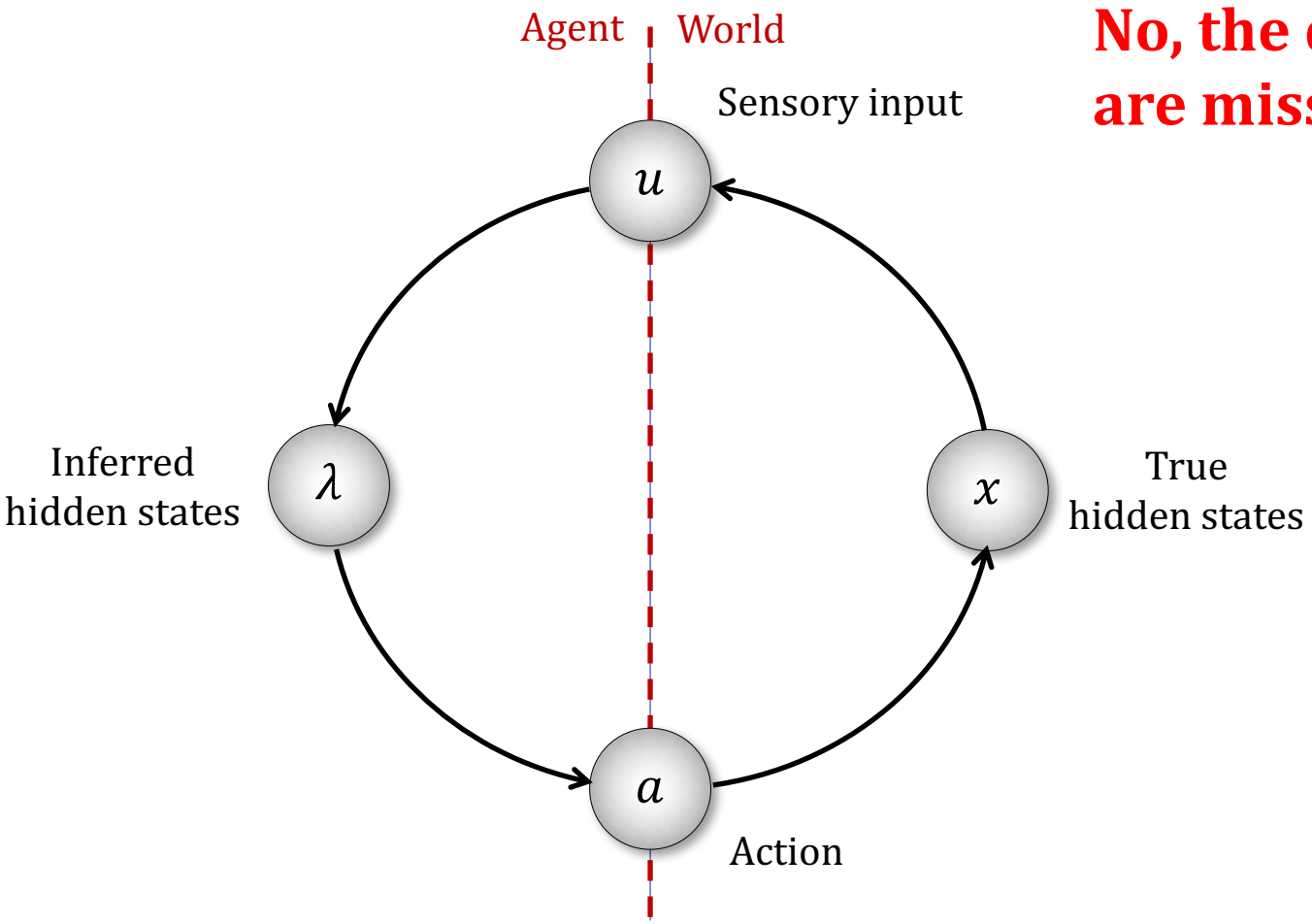
- Derive update equations

- Example: HGF

Neurobiology: predictive coding (e.g., Friston, 2005)



But: does inference as we've described it adequately describe the situation of actual biological agents?



No, the dynamics are missing!

What about dynamics?

Up to now, we've only looked at inference on static quantities, but biological agents live in a continually changing world.

In our example, the boat's position changes and with it the angle to the lighthouse.

How can we take into account that old information becomes obsolete? If we don't, our learning rate becomes smaller and smaller because our eqations were derived under the assumption that we're accumulating information about a stable quantity.

What's the simplest way to keep the learning rate from going too low?

Keep it constant!

So, taking the update equation for the mean of our observations as our point of departure...

$$\bar{x}_n = \bar{x}_{n-1} + \frac{1}{n}(x_n - \bar{x}_{n-1}),$$

... we simply replace $\frac{1}{n}$ with a constant α:

$$\mu_n = \mu_{n-1} + \alpha(x_n - \mu_{n-1}).$$

This is called *Rescorla-Wagner learning* [although it wasn't this line of reasoning that led Rescorla & Wagner (1972) to their formulation].

Does a constant learning rate solve our problems?

Partly: it implies a certain rate of forgetting because it amounts to taking only the $n = \frac{1}{\alpha}$ last data points into account. But...

... if the learning rate is supposed to reflect uncertainty in Bayesian inference, then how do we

(a) know that α reflects the right level of uncertainty at any one time, and

(b) account for changes in uncertainty if α is constant?

What we really need is an adaptive learning that accurately reflects uncertainty.

Needed: an adaptive learning rate that accurately reflects uncertainty

This requires us to think a bit more about what kinds of uncertainty we are dealing with.

A possible taxonomy of uncertainty is (cf. Yu & Dayan, 2003; Payzan-LeNestour & Bossaerts, 2011):

(a) **outcome uncertainty** that remains unaccounted for by the model, called *risk* by economists (π_ε in our Bayesian example); this uncertainty remains even when we know all parameters exactly,

(b) **informational** or *expected* uncertainty about the value of model parameters ($\pi_{\vartheta|x}$ in the Bayesian example),

(c) **environmental** or *unexpected* uncertainty owing to changes in model parameters (not accounted for in our Bayesian example, hence unexpected).

An adaptive learning rate that accurately reflects uncertainty

Various efforts have been made to come up with an adaptive learning rate:

- Kalman (1960)
- Sutton (1992)
- Nassar et al. (2010)
- Payzan-LeNestour & Bossaerts (2011)
- Mathys et al. (2011)
- Wilson et al. (2013)

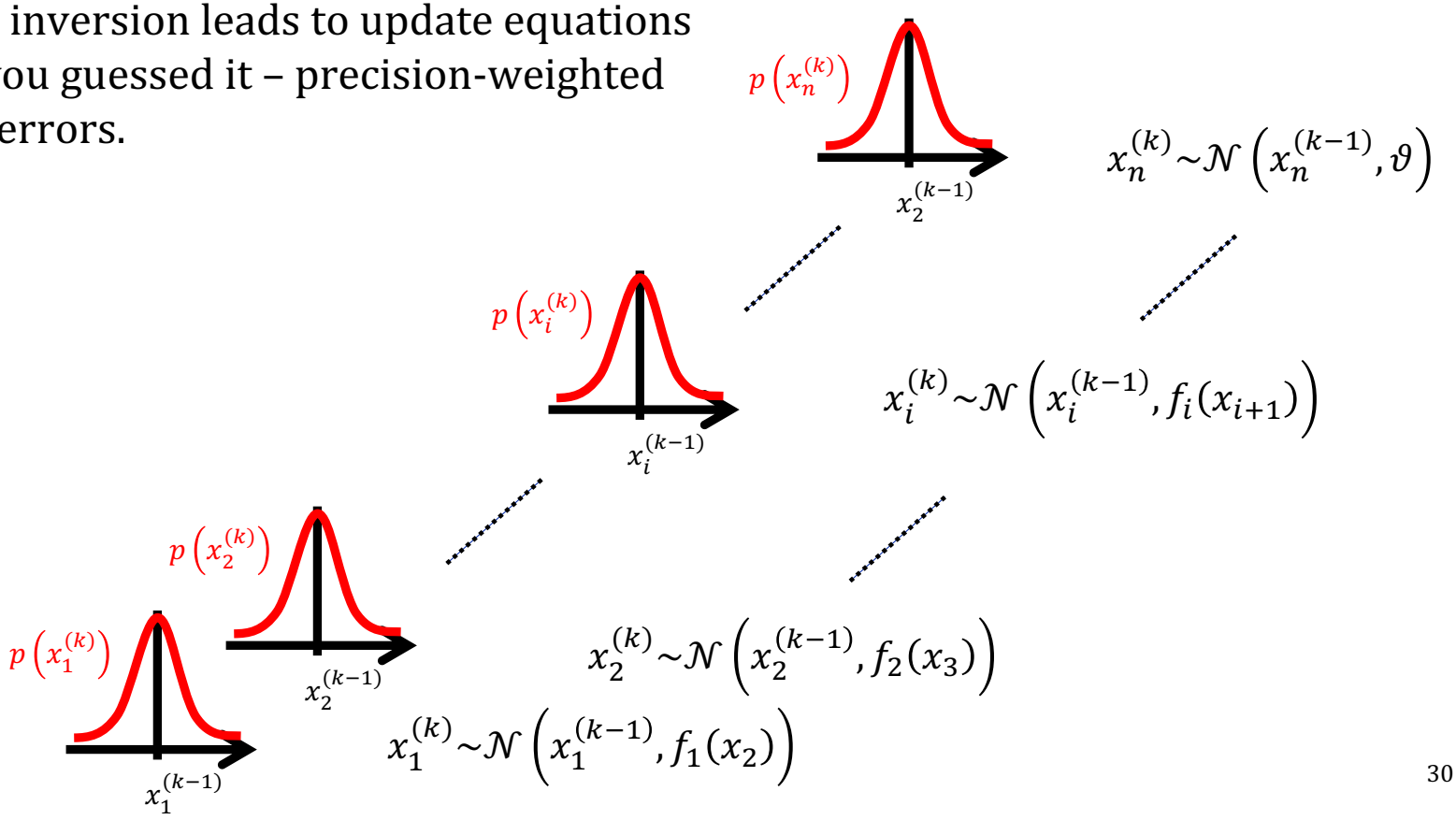
The Kalman filter is optimal for linear dynamical systems, but realistic data usually require non-linear models.

Mathys et al. use a generic non-linear hierarchical Bayesian model that allows us to derive update equations that are optimal in the sense that they minimize surprise.

The hierarchical Gaussian filter (HGF, Mathys et al., 2011; 2014)

The HGF provides a generic solution to the problem of adapting one's learning rate in a volatile environment.

Variational inversion leads to update equations that are – you guessed it – precision-weighted prediction errors.



Coupling between levels

Since f has to be everywhere positive, we cannot approximate it by expanding in powers. Instead, we expand its logarithm.

$$f(x) > 0 \; \forall \, x \implies \exists \, g : f(x) = \exp\big(g(x)\big) \; \forall \, x$$

$$g(x) = g(a) + g'(a) \cdot (x - a) + O(2) = \log f(x) =$$

$$= \log f(a) + \frac{f'(a)}{f(a)} \cdot (x - a) + O(2) =$$

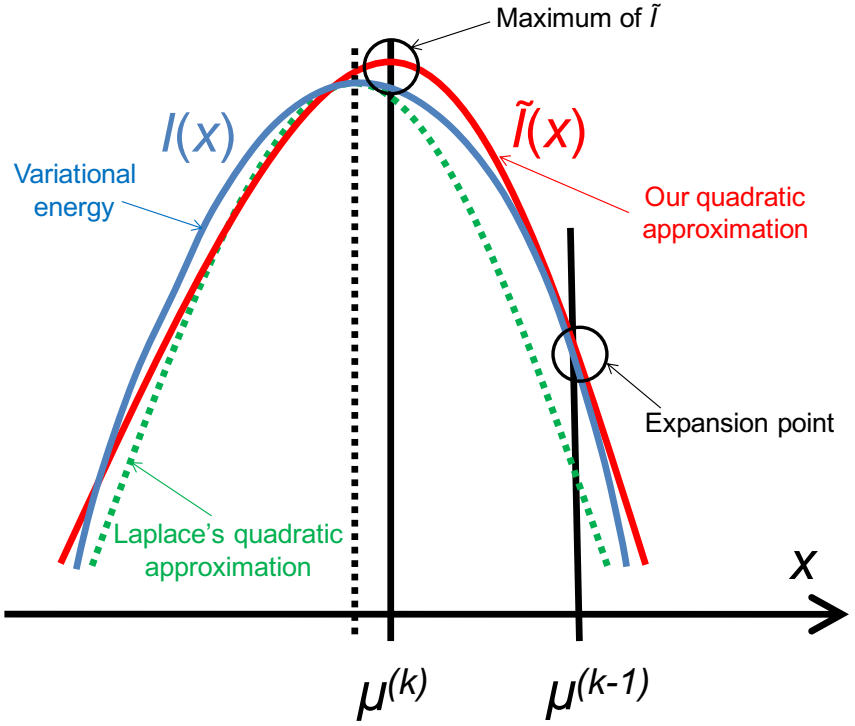
$$= \underbrace{\frac{f'(a)}{f(a)}}_{\stackrel{\text{def}}{=} \kappa} \cdot x + \underbrace{\log f(a) - a \cdot \frac{f'(a)}{f(a)}}_{\stackrel{\text{def}}{=} \omega} + O(2) =$$

$$= \kappa x + \omega + O(2)$$

$$\implies f(x) \approx \exp(\kappa x + \omega)$$

Variational inversion

- A quadratic approximation is found by expanding to second order about the expectation $\mu^{(k-1)}$.
- The update in the sufficient statistics of the approximate posterior is then performed by analytically finding the maximum of the quadratic approximation.



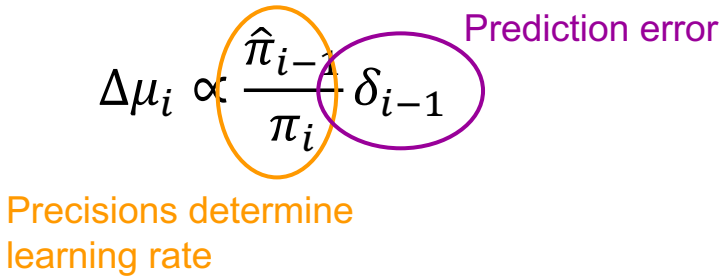
$$\sigma_i^{(k)} = -\frac{1}{\partial^2 I\left(\mu_i^{(k-1)}\right)}$$

$$\mu_i^{(k)} = \mu_i^{(k-1)} - \frac{\partial I\left(\mu_i^{(k-1)}\right)}{\partial^2 I\left(\mu_i^{(k-1)}\right)} = \mu_i^{(k-1)} + \sigma_i^{(k)} \partial I\left(\mu_i^{(k-1)}\right)$$

Mathys et al. (2011). *Front. Hum. Neurosci.*, 5:39.

Variational inversion and update equations

- Inversion proceeds by introducing a mean field approximation and fitting quadratic approximations to the resulting variational energies (Mathys et al., 2011).
- This leads to **simple one-step update equations** for the sufficient statistics (mean and precision) of the approximate Gaussian posteriors of the states x_i.
- The updates of the means have the same structure as value updates in Rescorla-Wagner learning:



- Furthermore, the updates are **precision-weighted prediction errors**.

Precision-weighting of volatility updates

Comparison to the simple non-hierarchical Bayesian update:

HGF: $$\mu_i^{(k)} = \mu_i^{(k-1)} + \frac{1}{2}\kappa_{i-1}\, v_{i-1}^{(k)} \cdot \frac{\hat{\pi}_{i-1}^{(k)}}{\pi_i^{(k)}} \cdot \delta_{i-1}^{(k)}$$

Precision-weighted prediction error

Simple Gaussian: $$\mu_{\vartheta|x} = \mu_\vartheta + \frac{\pi_\varepsilon}{\pi_{\vartheta|x}}(x - \mu_\vartheta)$$

Updates at the outcome level

At the outcome level (i.e., at the very bottom of the hierarchy), we have

$$u^{(k)} \sim \mathcal{N}\left(x_1^{(k)}, \hat{\pi}_u^{-1}\right)$$

This gives us the following update for our belief on x_1 (our quantity of interest):

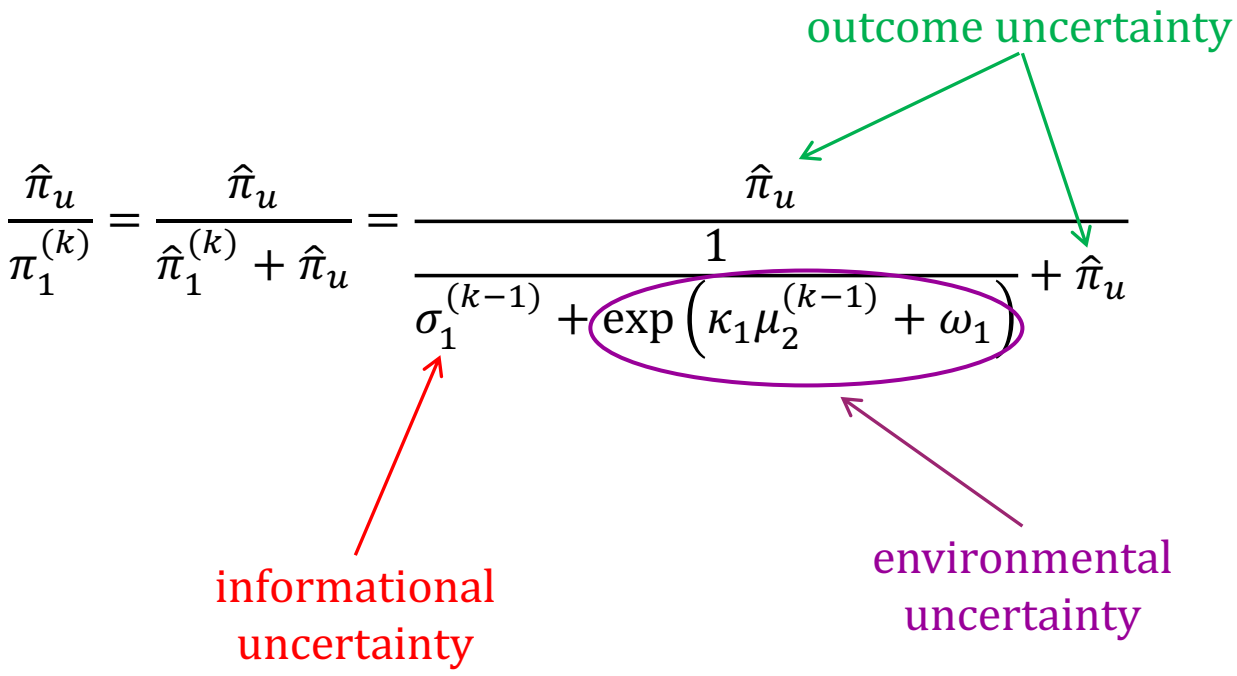
$$\pi_1^{(k)} = \hat{\pi}_1^{(k)} + \hat{\pi}_u$$

$$\mu_1^{(k)} = \mu_1^{(k-1)} + \frac{\hat{\pi}_u}{\pi_1^{(k)}}\left(u^{(k)} - \mu_1^{(k-1)}\right)$$

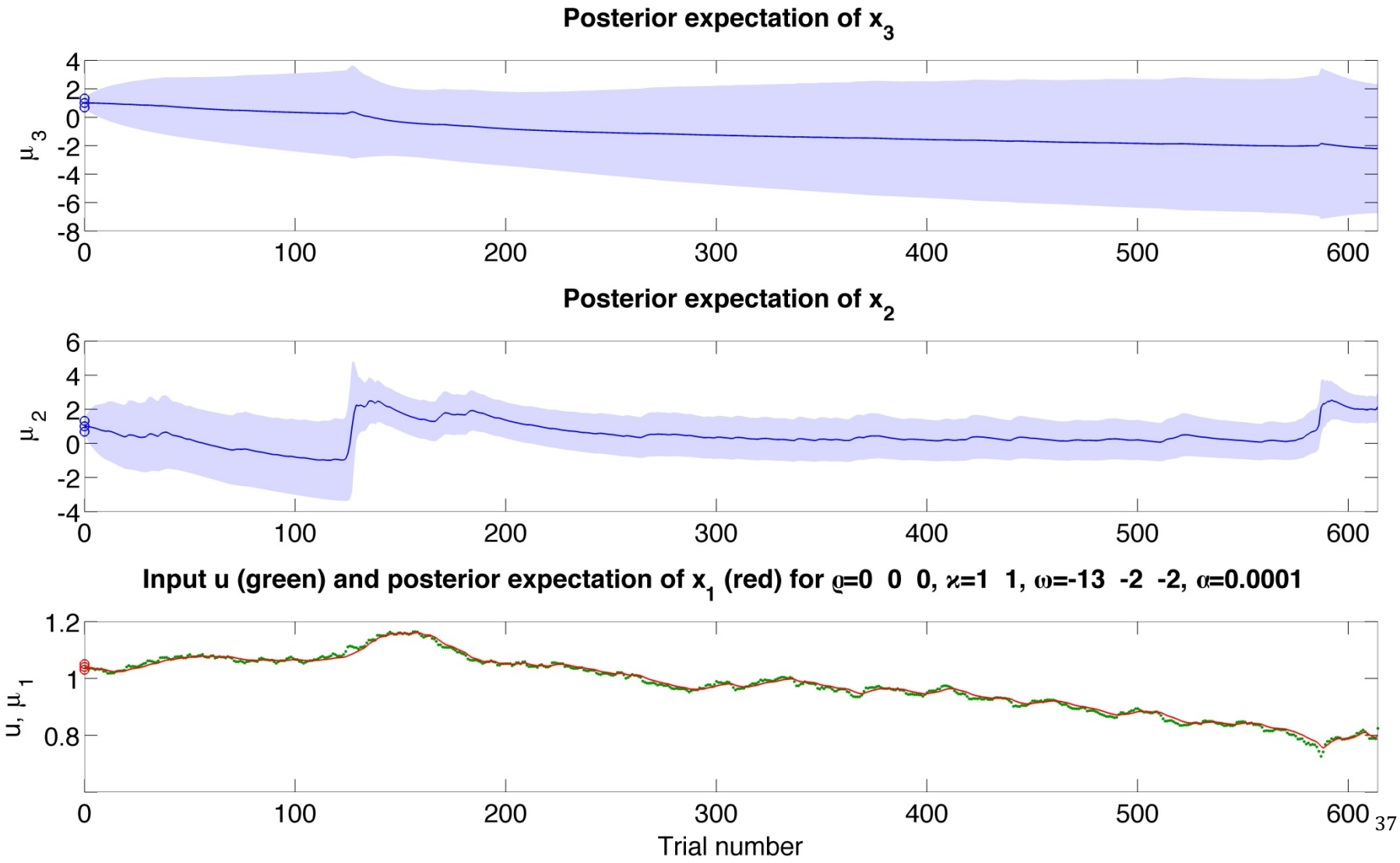
The familiar structure again – but now with a learning rate that is responsive to all kinds of uncertainty, including environmental (unexpected) uncertainty.

The learning rate in the HGF

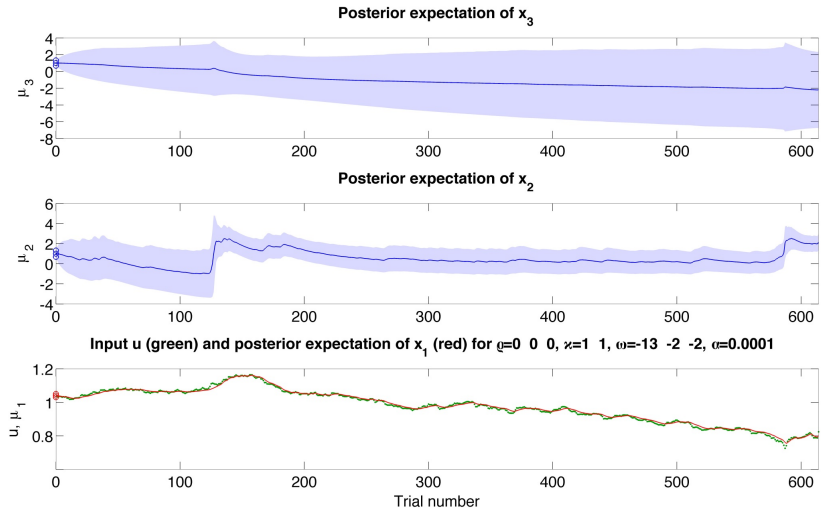
Unpacking the learning rate, we see:

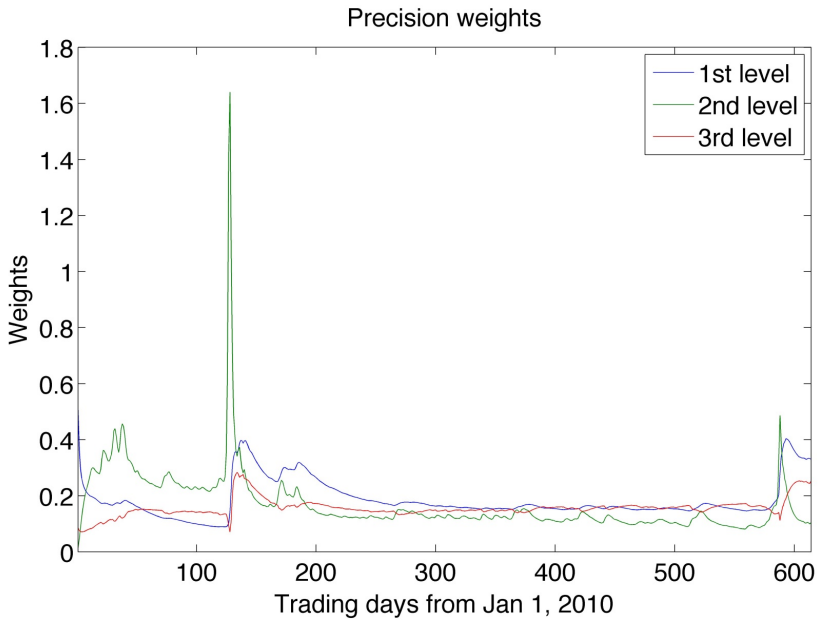


3-level HGF for continuous observations



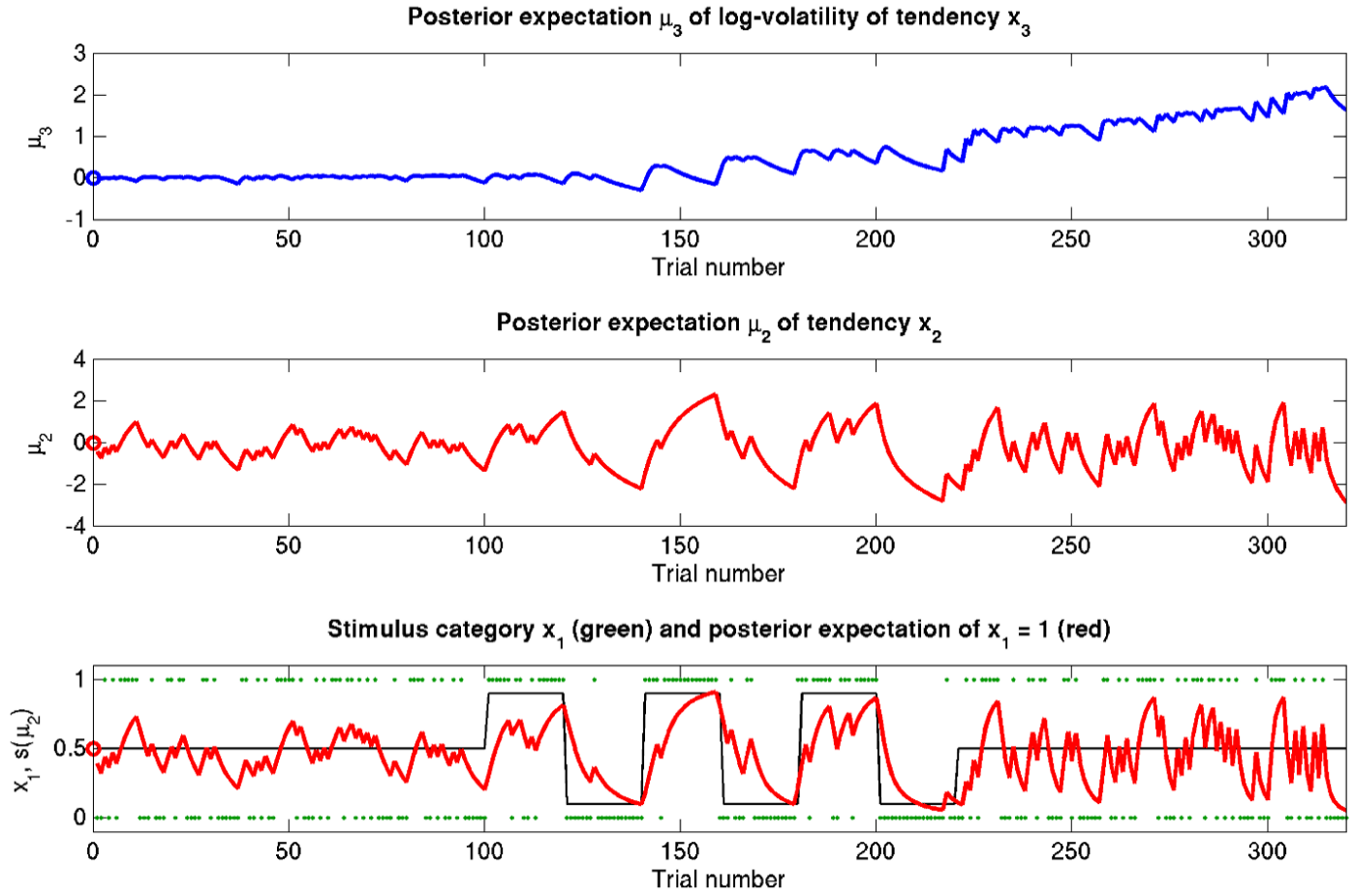
Example of precision weight trajectory



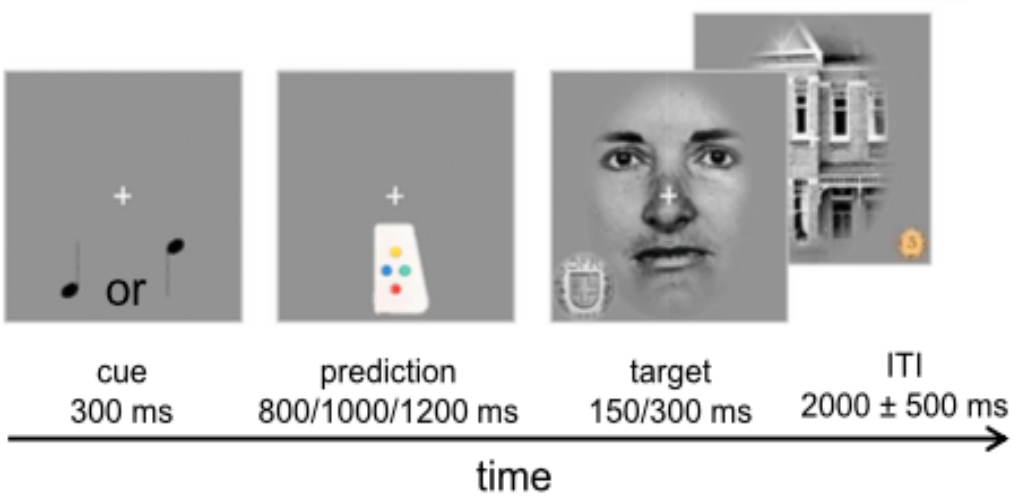


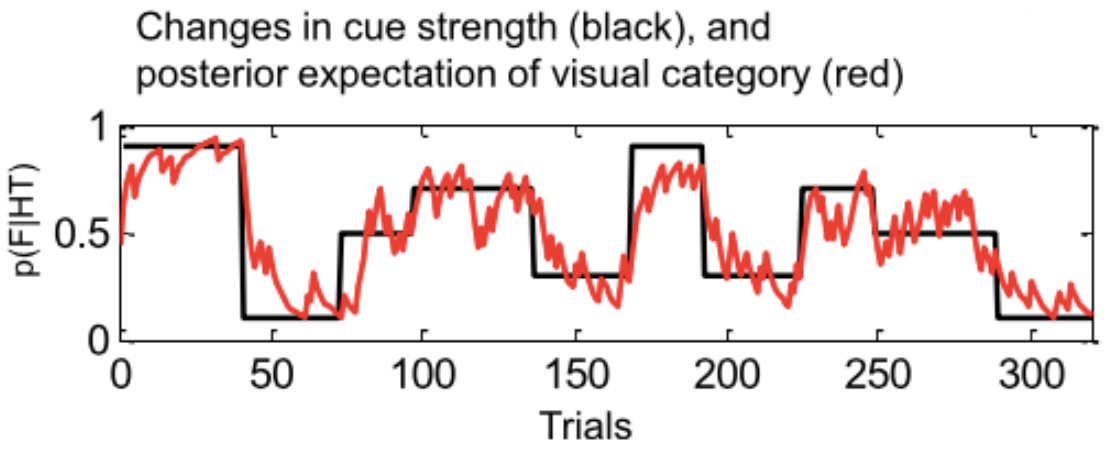
Context effects on the learning rate

Simulation: $\vartheta = 0.5,\ \omega = -2.2,\ \kappa = 1.4$



HGF: empirical evidence (Iglesias et al, Neuron, 2013)





HGF: empirical evidence (Iglesias et al, Neuron, 2013)

Model comparison:

	Behavioral study		fMRI study 1		fMRI study 2	
BMS results	PP	XP	PP	XP	PP	XP
HGF1	0.8435	1	0.7422	1	0.7166	1
HGF2	0.0259	0	0.0200	0	-	-
HGF3	0.0361	0	0.1404	0	0.1304	0
Sutton	0.0685	0	0.0710	0	0.0761	0
RW	0.0260	0	0.0264	0	0.0769	0

HGF: empirical evidence (Iglesias et al, Neuron, 2013)

Model comparison:

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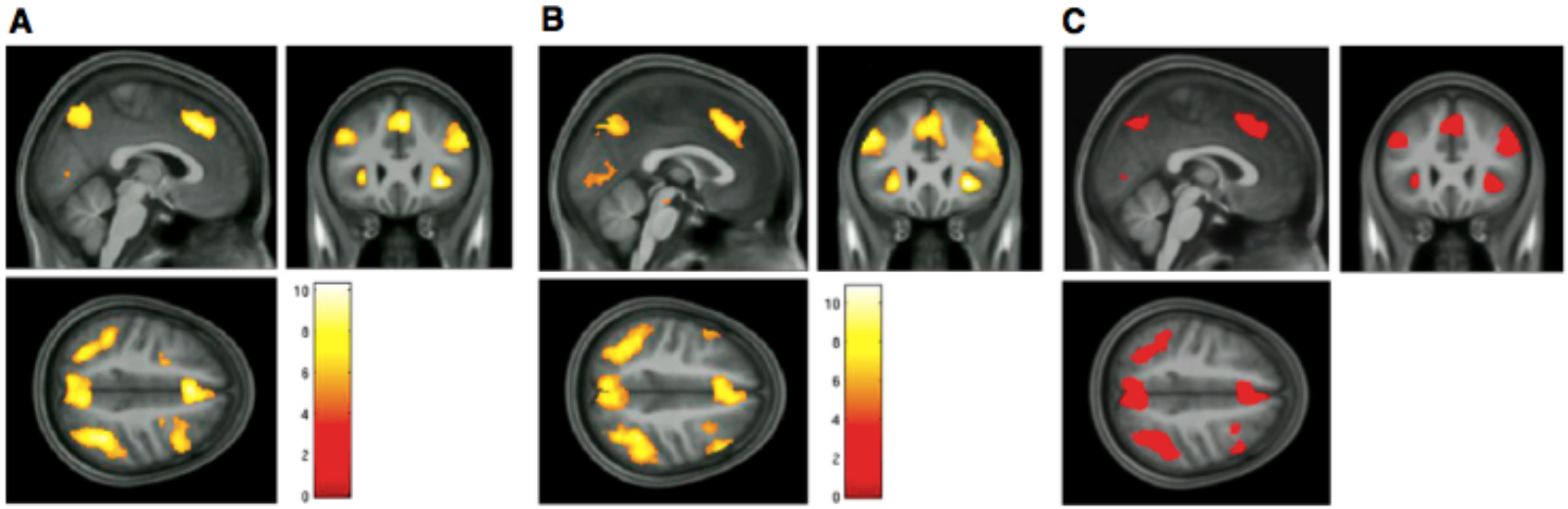


Figure 2. Whole-Brain Activations by ε_2

Activations by precision-weighted prediction error about visual stimulus outcome, ε_2, in the first fMRI study (A) and the second fMRI study (B). Both activation maps are shown at a threshold of $p < 0.05$, FWE corrected for multiple comparisons across the whole brain. To highlight replication across studies, (C) shows the results of a "logical AND" conjunction, illustrating voxels that were significantly activated in both studies.

HGF: empirical evidence (Iglesias et al, Neuron, 2013)

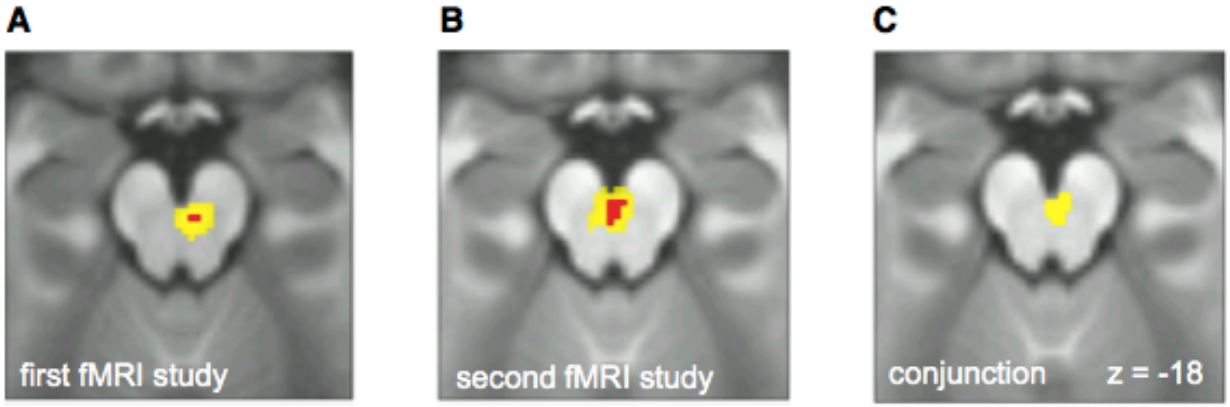


Figure 3. Midbrain Activation by ε_2
Activation of the dopaminergic VTA/SN associated with precision-weighted prediction error about stimulus category, ε_2. This activation is shown both at $p < 0.05$ FWE whole-brain corrected (red) and $p < 0.05$ FWE corrected for the volume of our anatomical mask comprising both dopaminergic and cholinergic nuclei (yellow).
(A) Results from the first fMRI study.
(B) Second fMRI study.
(C) Conjunction (logical AND) across both studies.

HGF: empirical evidence (Iglesias et al, Neuron, 2013)

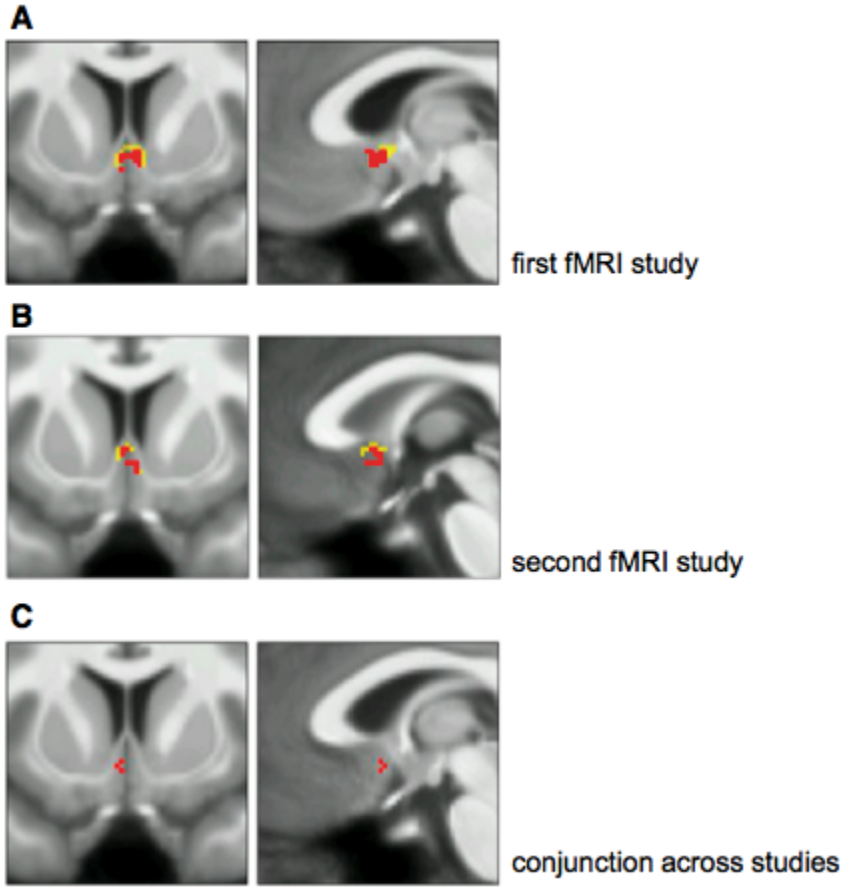
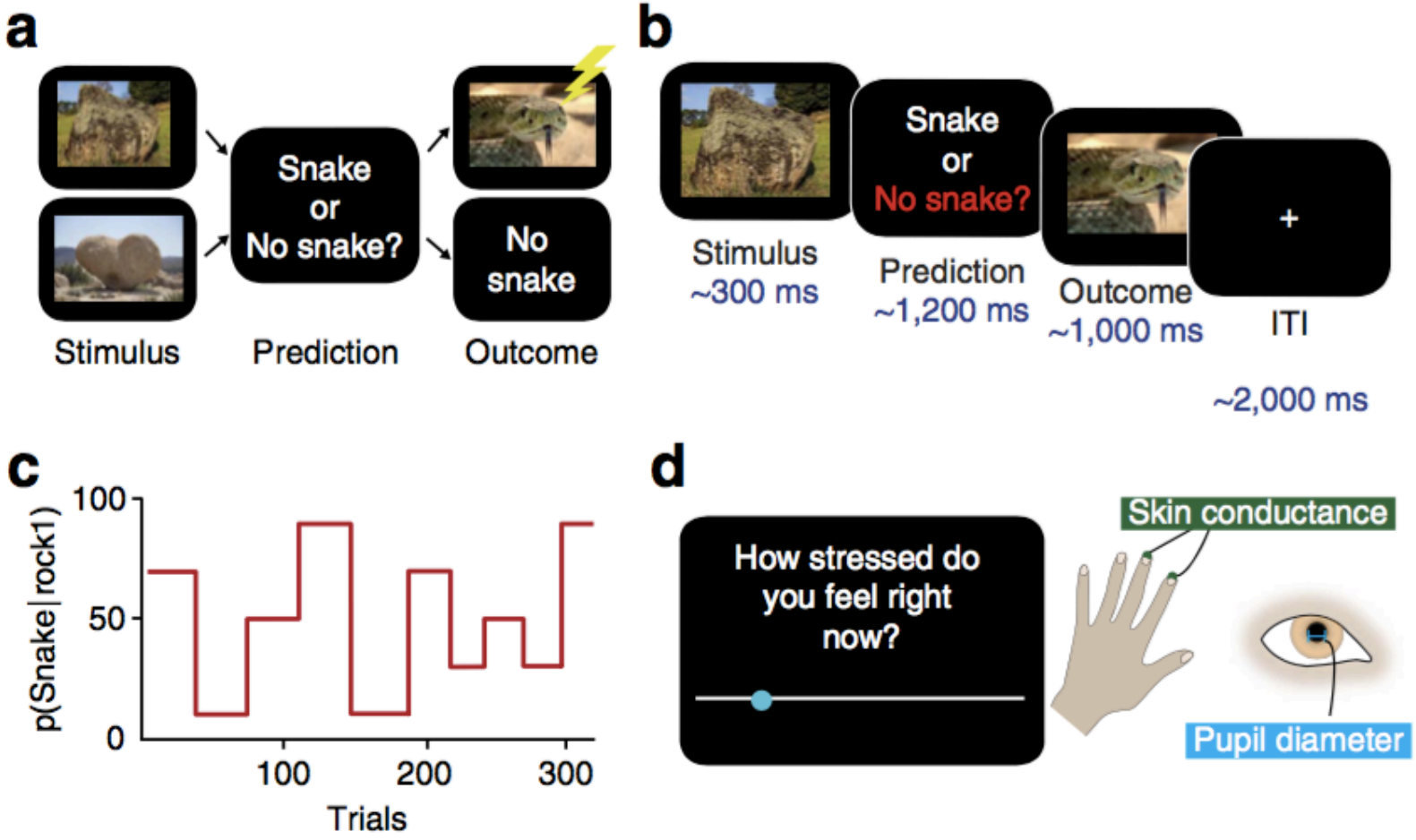


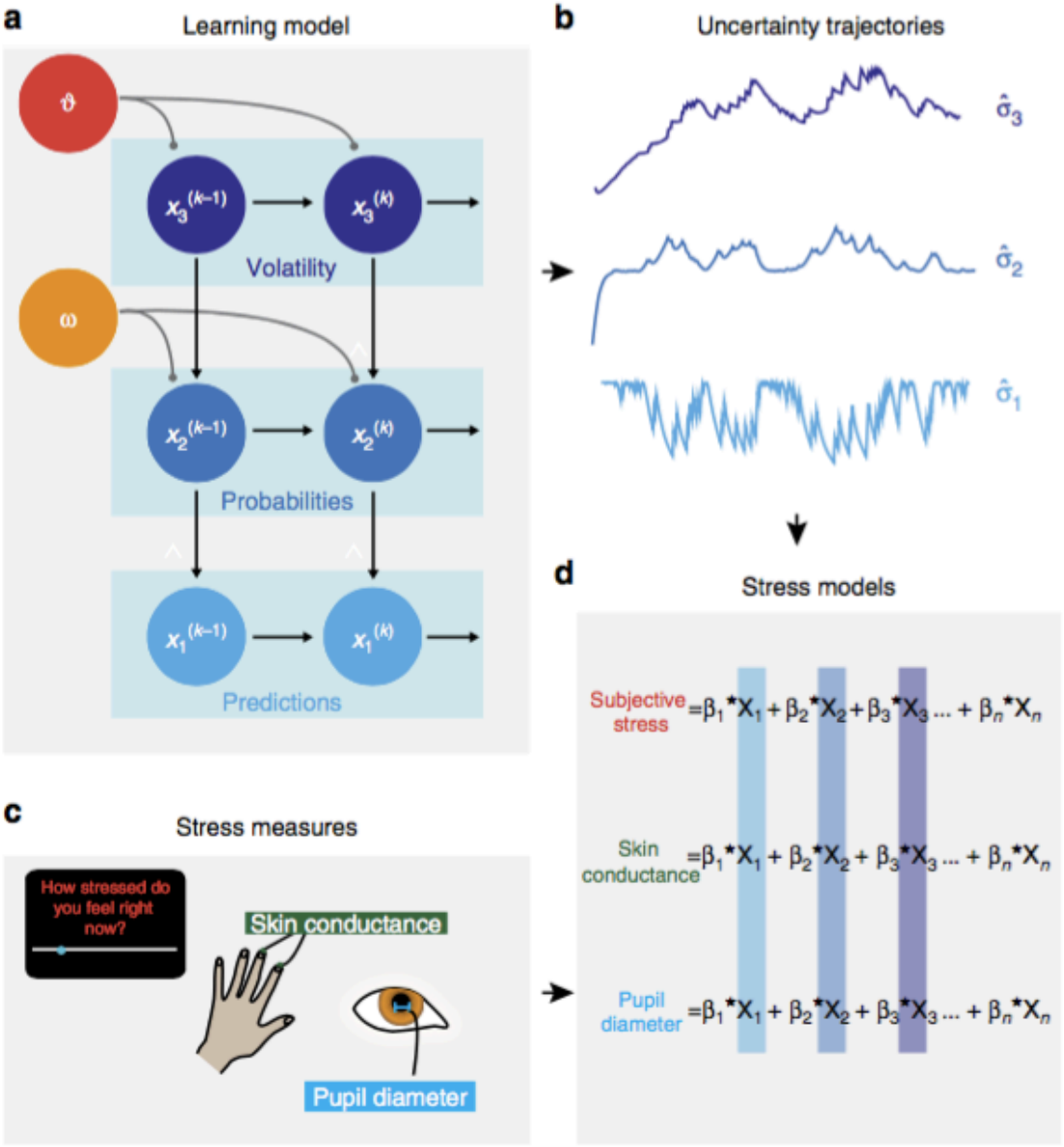
Figure 6. Basal Forebrain Activations by ε_3

Activation of the cholinergic basal forebrain associated with precision-weighted prediction error about stimulus probabilities ε_3 within the anatomically defined mask. For visualization of the activation area we overlay the results thresholded at $p < 0.05$ FWE corrected for the entire anatomical mask (red) on the results thresholded at $p < 0.001$ uncorrected (yellow) in the first (A: $x = 3, y = 9, z = -8$) and the second fMRI study (B: $x = 0, y = 10, z = -8$). (C) The conjunction analysis ("logical AND") across both studies ($x = 2, y = 11, z = -8$).

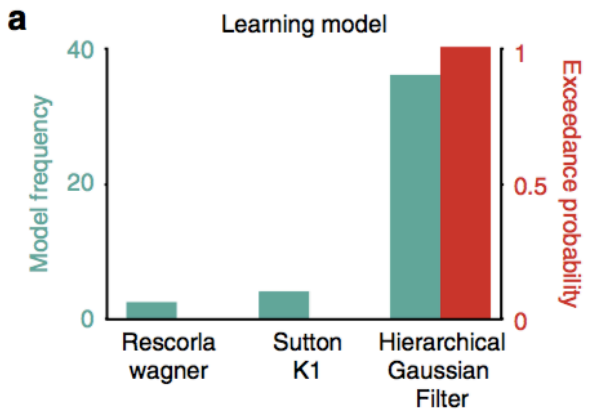
HGF: empirical evidence (De Berker et al, Nature Comms, 2016)

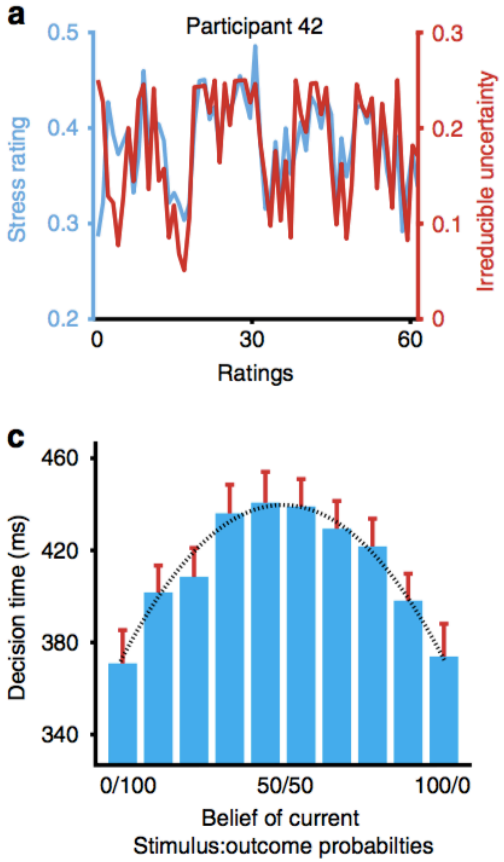


HGF: empirical evidence (De Berker et al, Nature Comms, 2016)

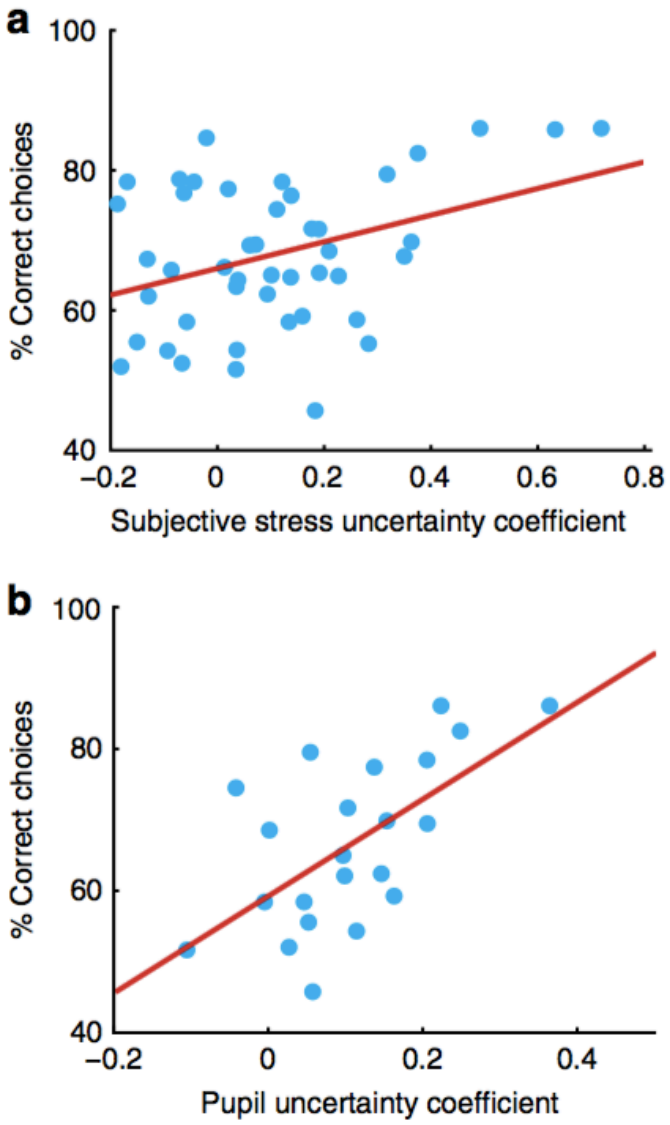


HGF: empirical evidence (De Berker et al, Nature Comms, 2016)

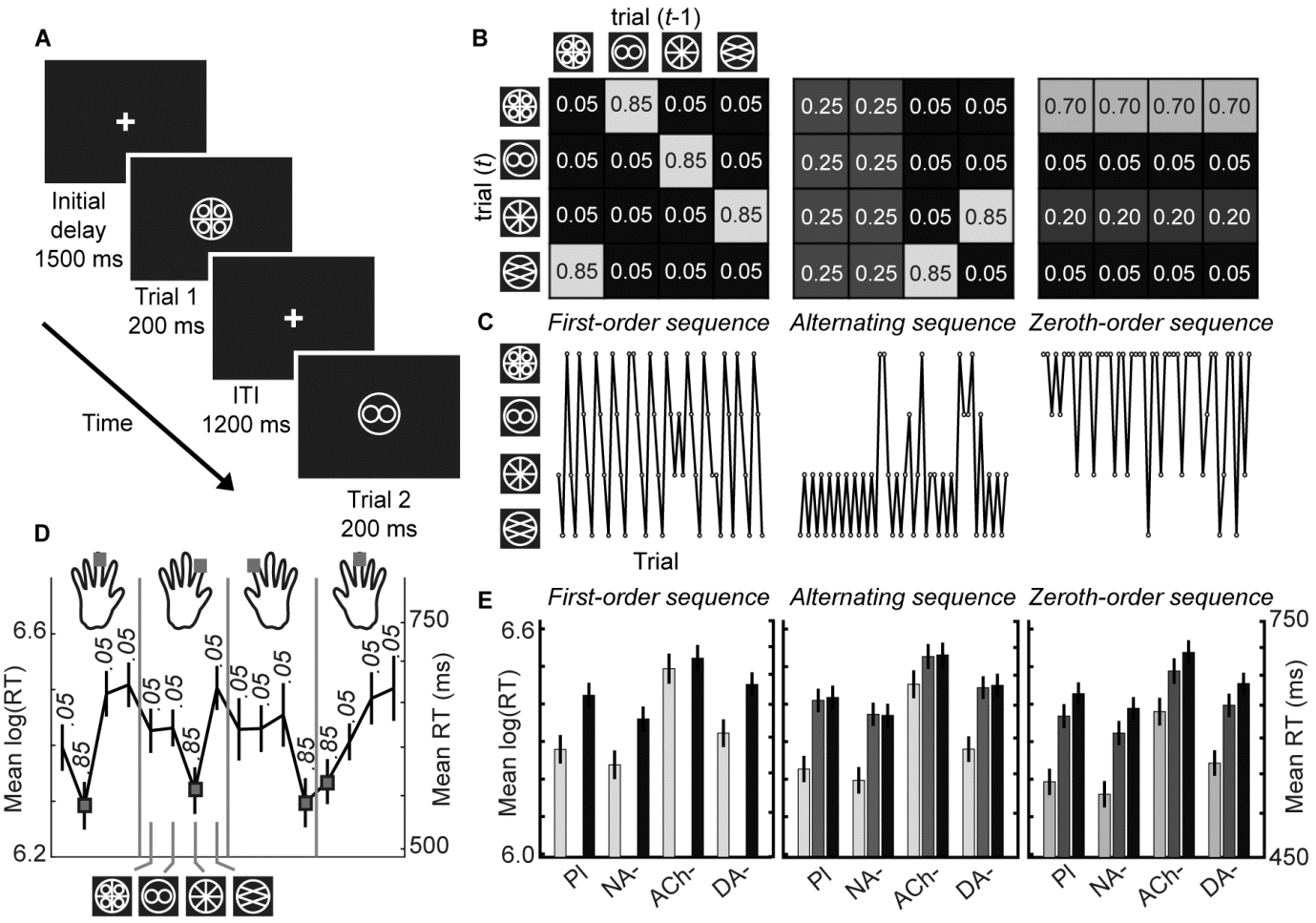




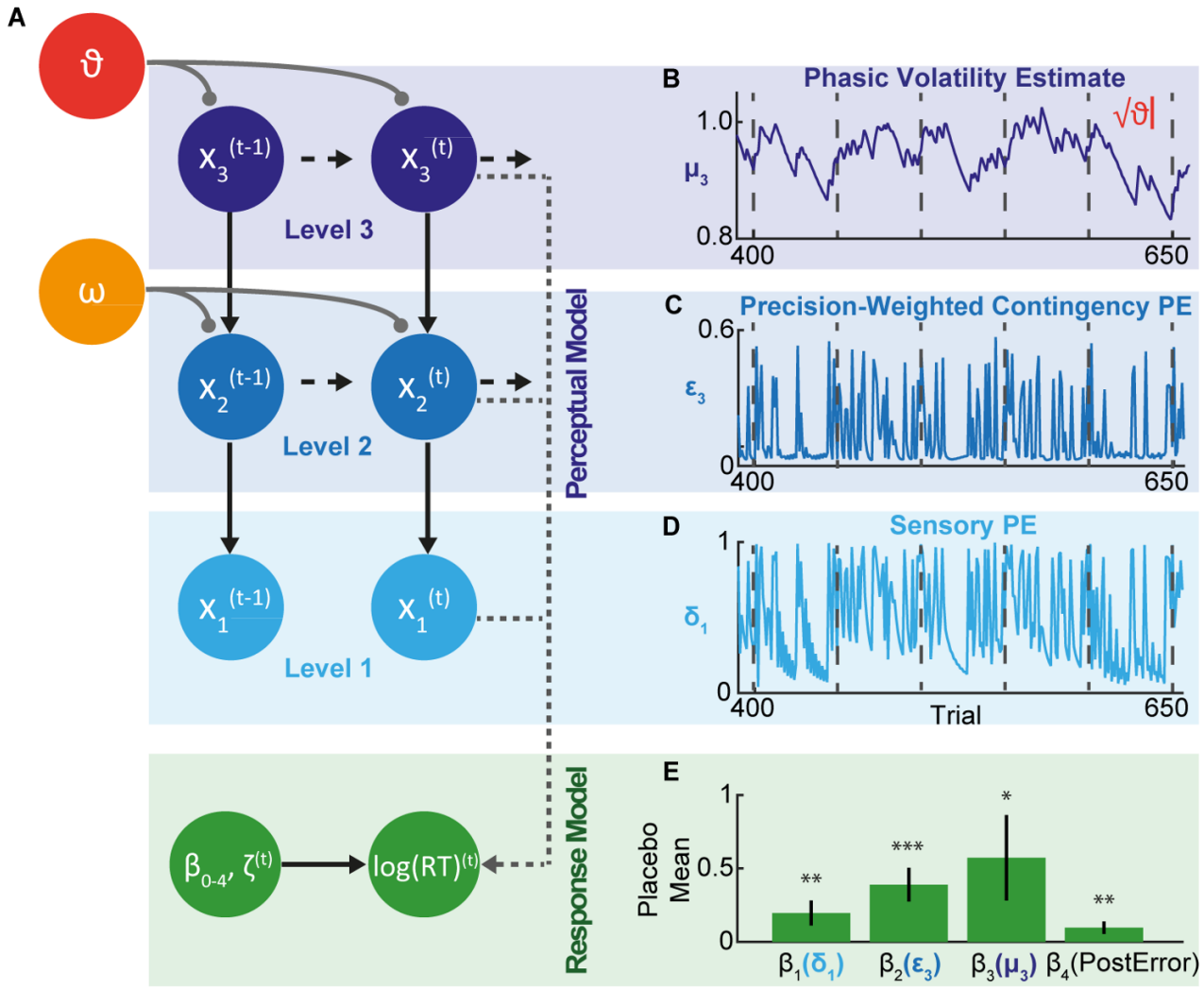
HGF: empirical evidence (De Berker et al, Nature Comms, 2016)



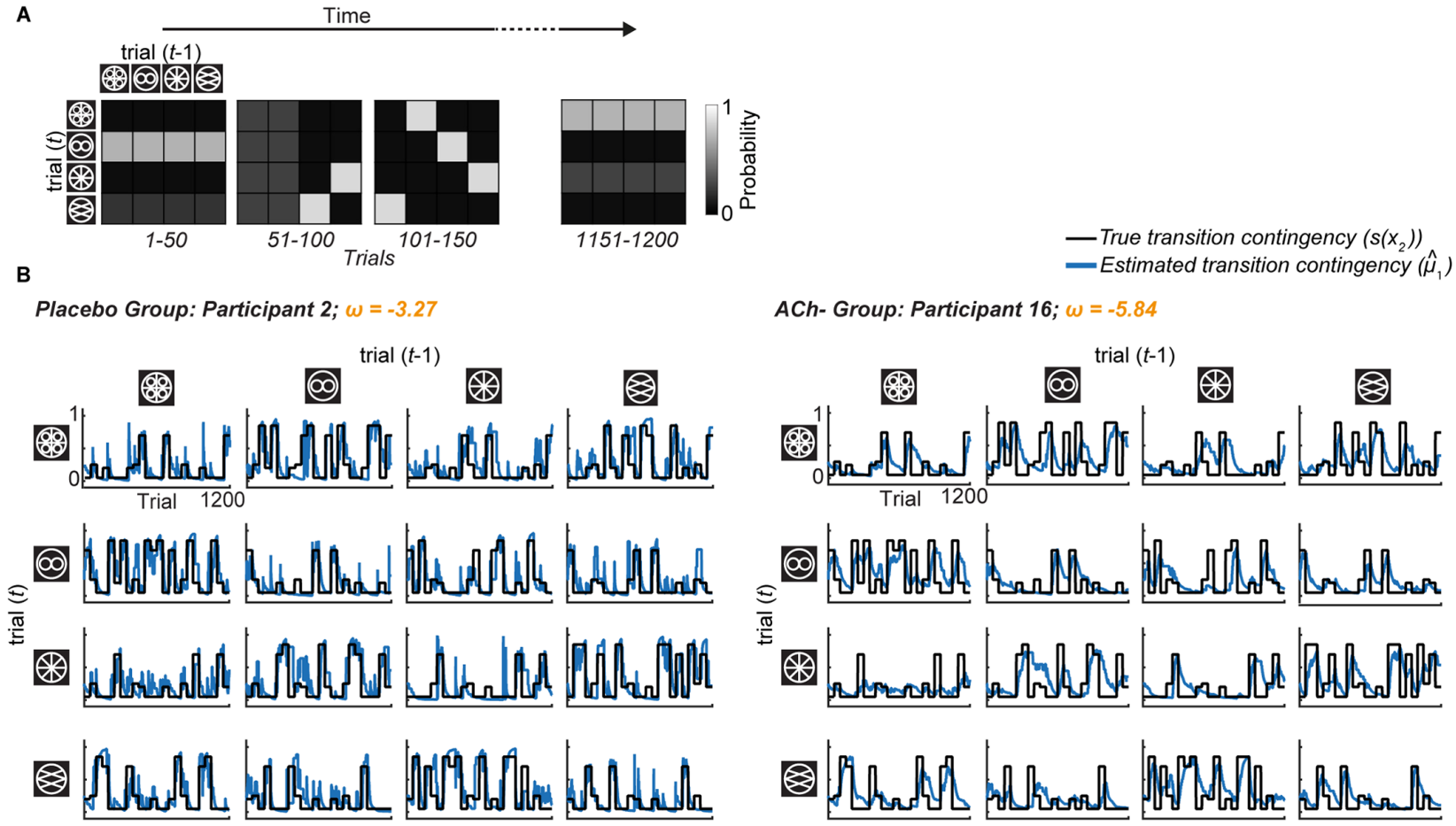
HGF: empirical evidence (Marshall & Mathys et al., PLOS Biol., 2016)



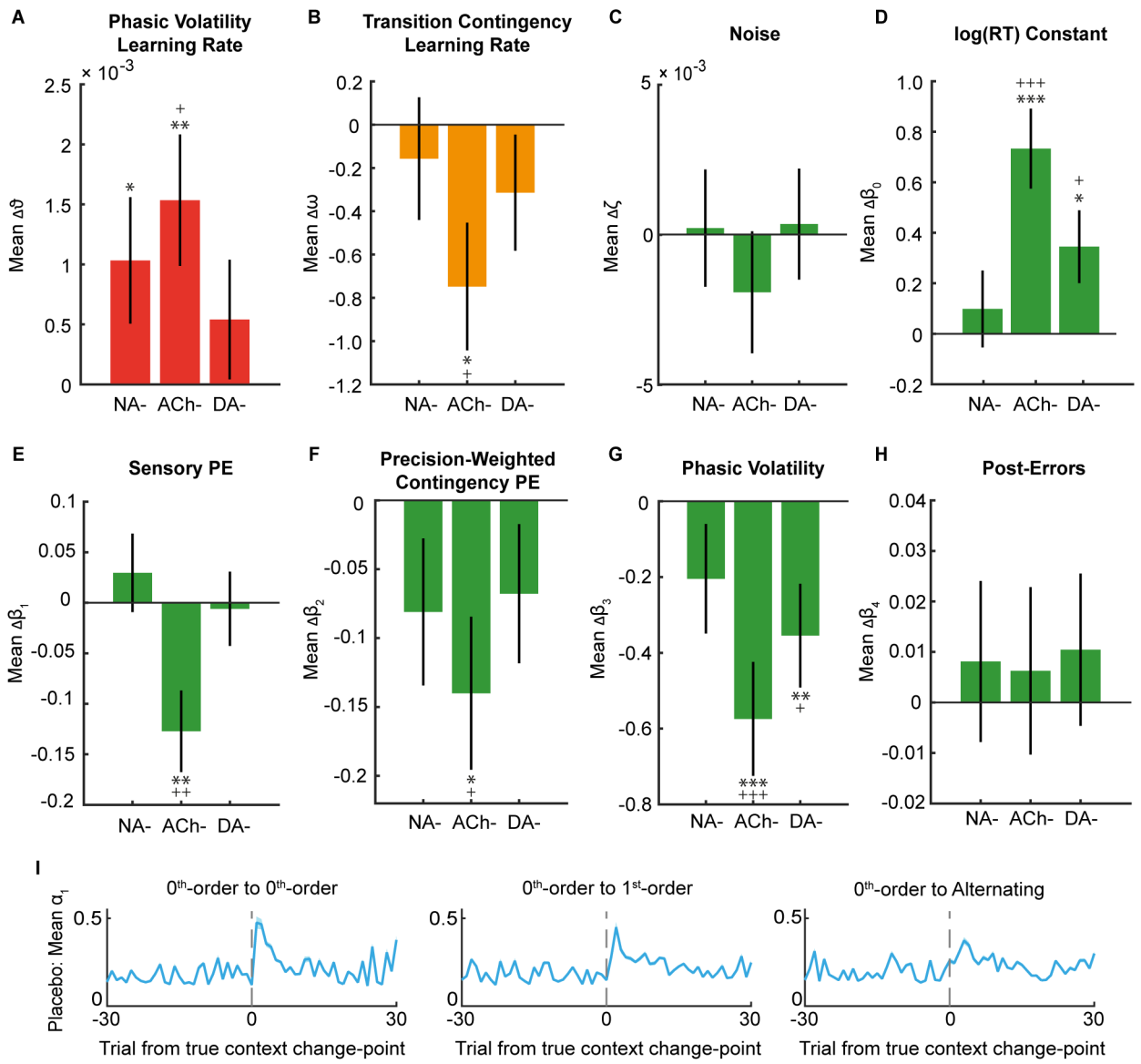
HGF: empirical evidence (Marshall & Mathys et al., PLOS Biol., 2016)



HGF: empirical evidence (Marshall & Mathys et al., PLOS Biol., 2016)



HGF: empirical evidence (Marshall & Mathys et al., PLOS Biol., 2016)



HGF: empirical evidence (Diaconescu et al., in preparation)

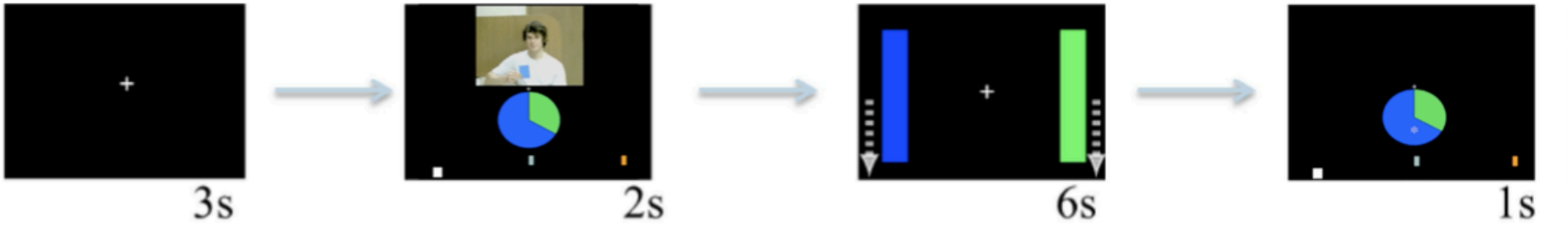


Fig. 1. Experimental Paradigm: 100 male volunteers played a binary lottery task and received advice about which option to choose from a more informed agent who was also incentivized to influence the participants' choices. To decide whether to take his advice into account, participants also inferred on the other's intentions and how they changed in time.

HGF: empirical evidence (Diaconescu et al., in preparation)

6. Volatility Precision

5. Volatility PE

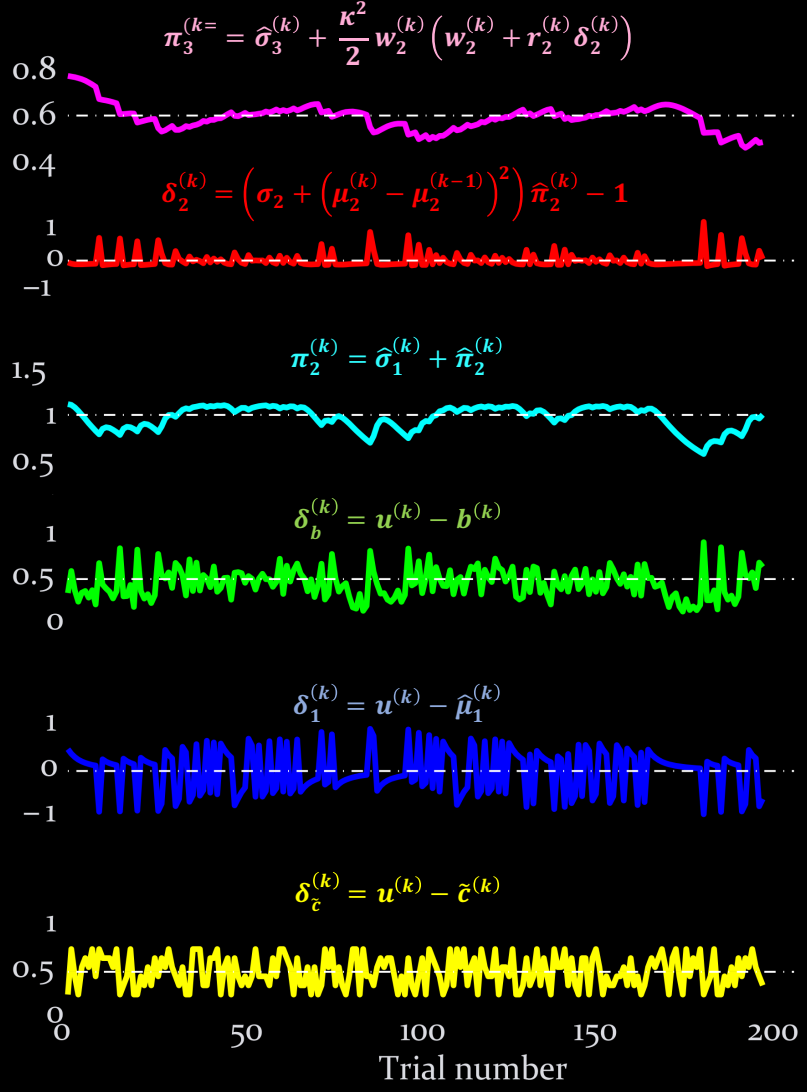
4. Advice Precision

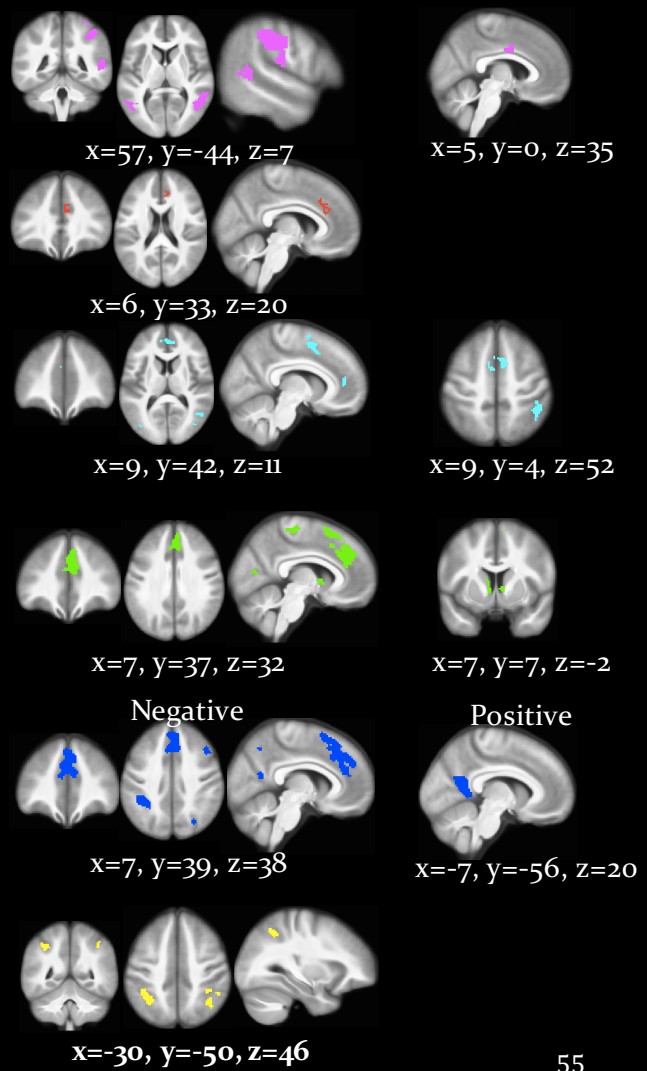
3. Outcome PE

2. Advice PE

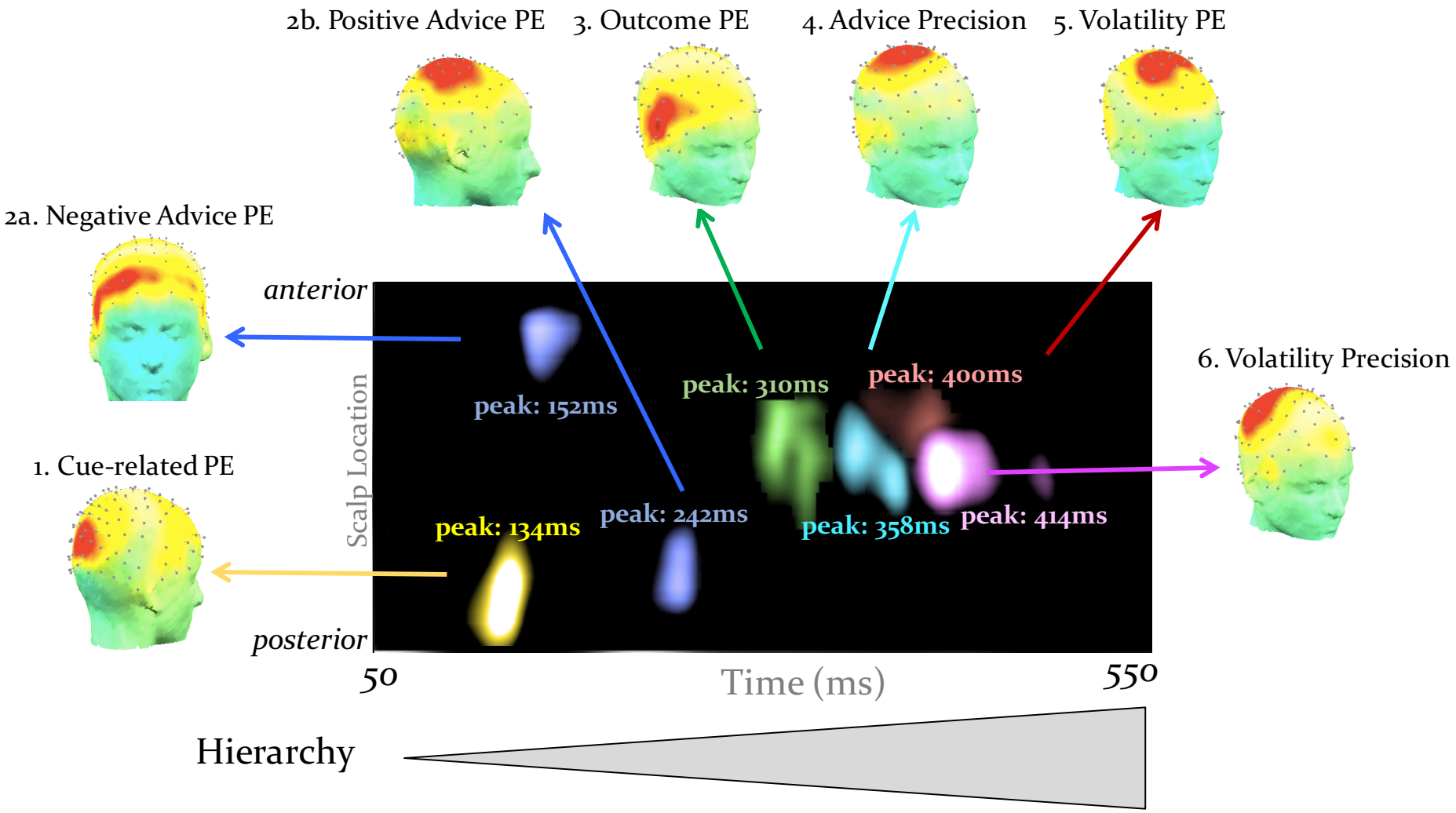
1. Cue-Related PE

Hierarchy

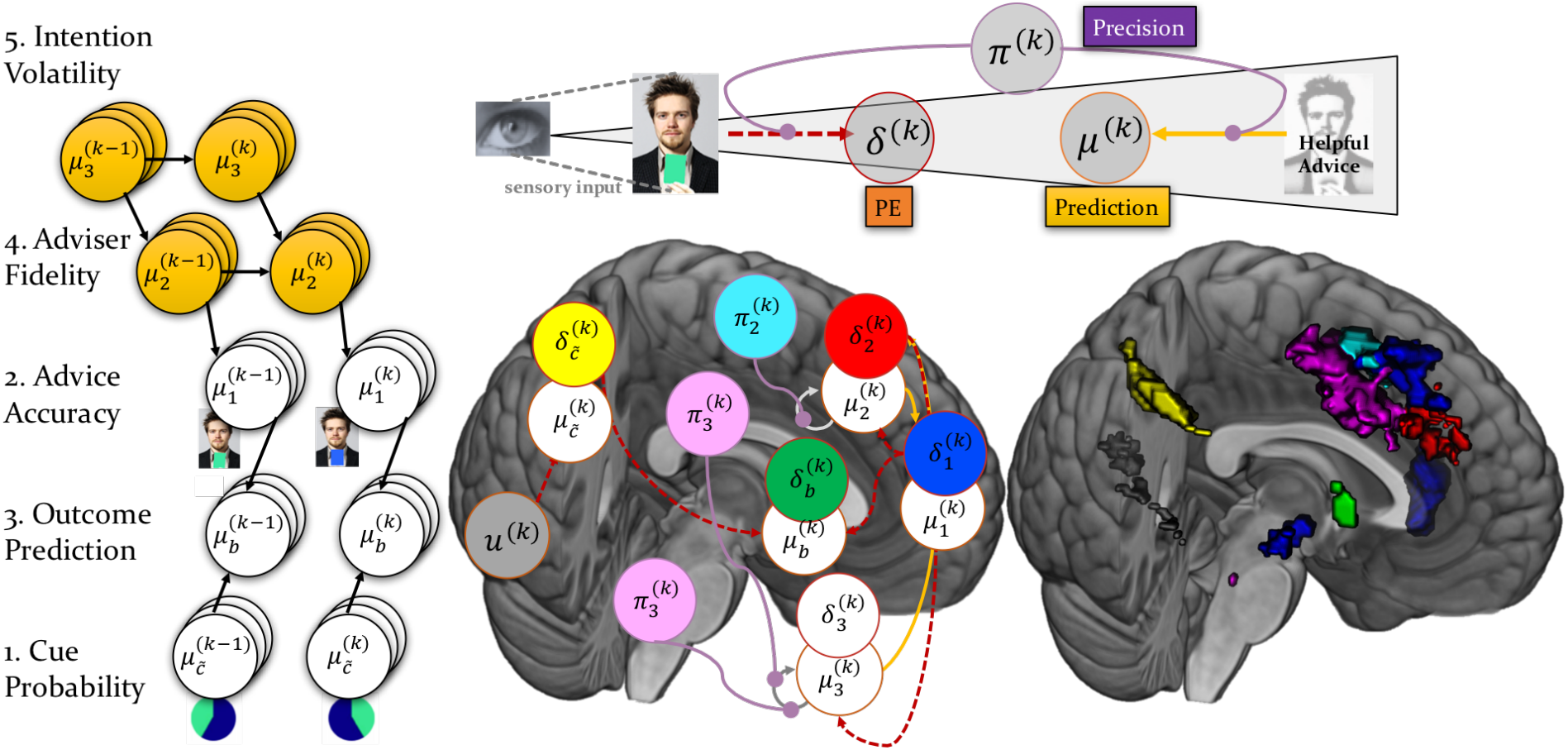




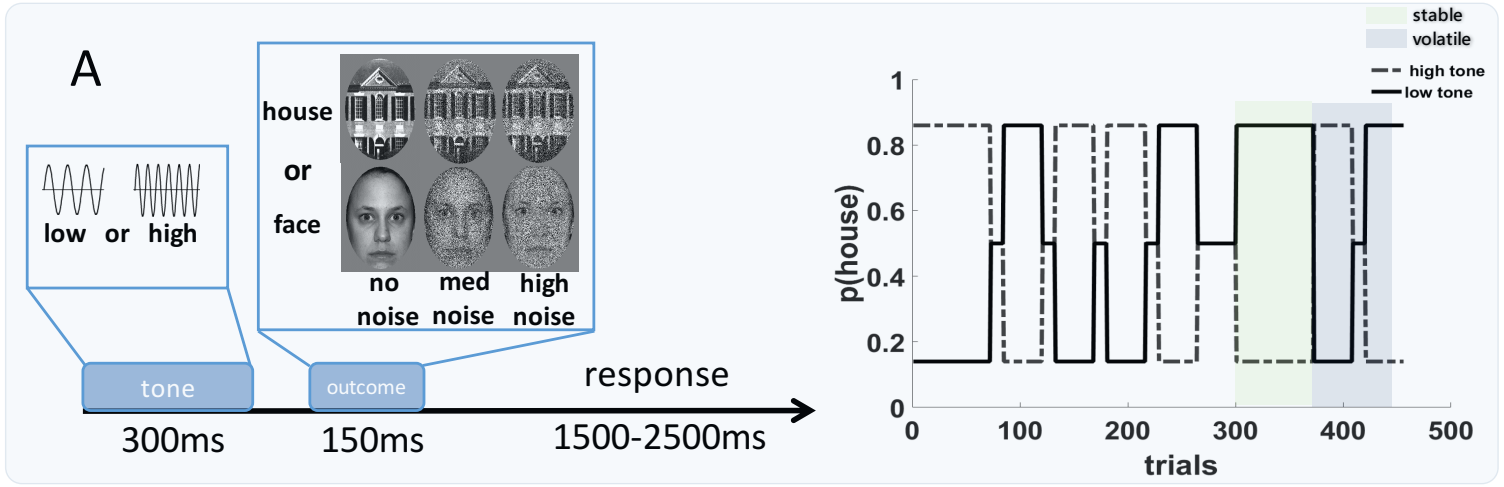
HGF: empirical evidence (Diaconescu et al., in preparation)

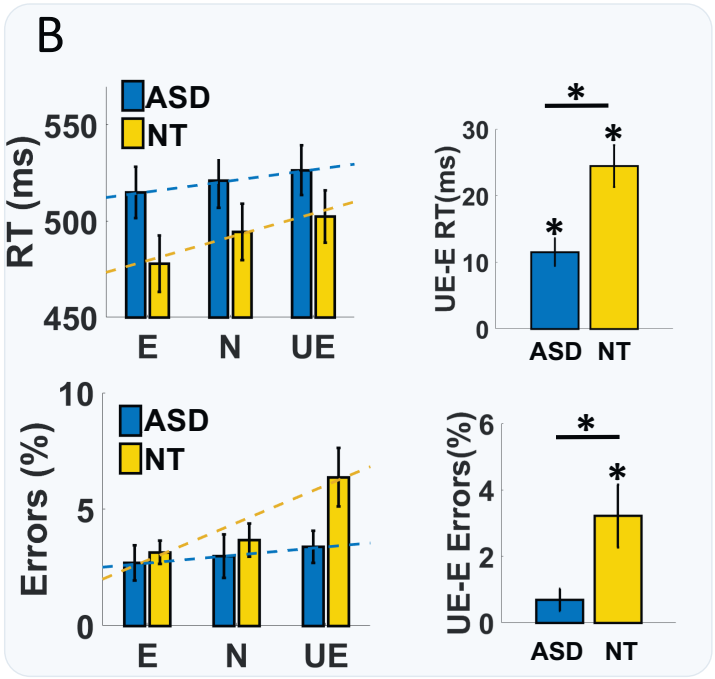


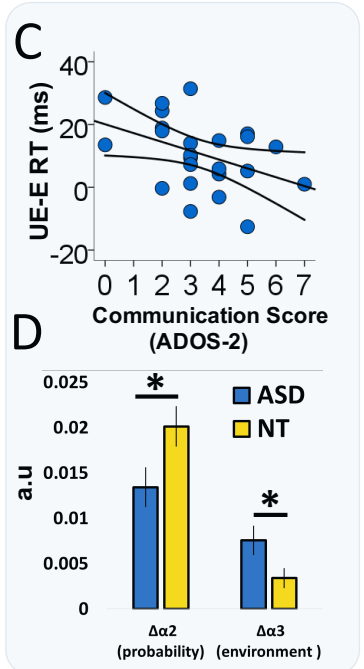
HGF: empirical evidence (Diaconescu et al., in preparation)

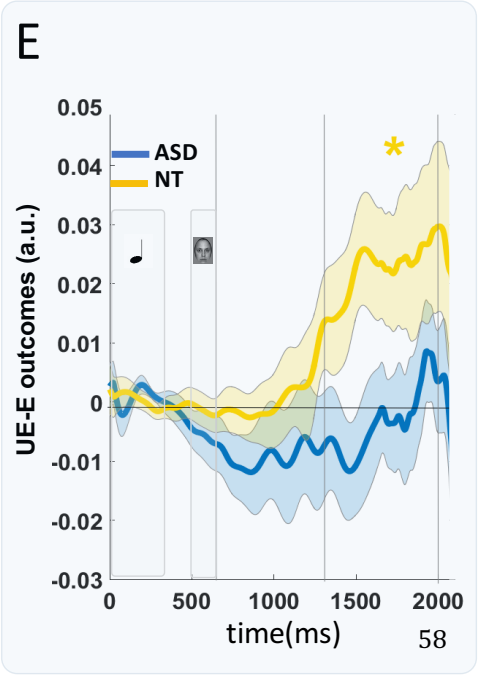


HGF: empirical evidence (Lawson et al., in revision)



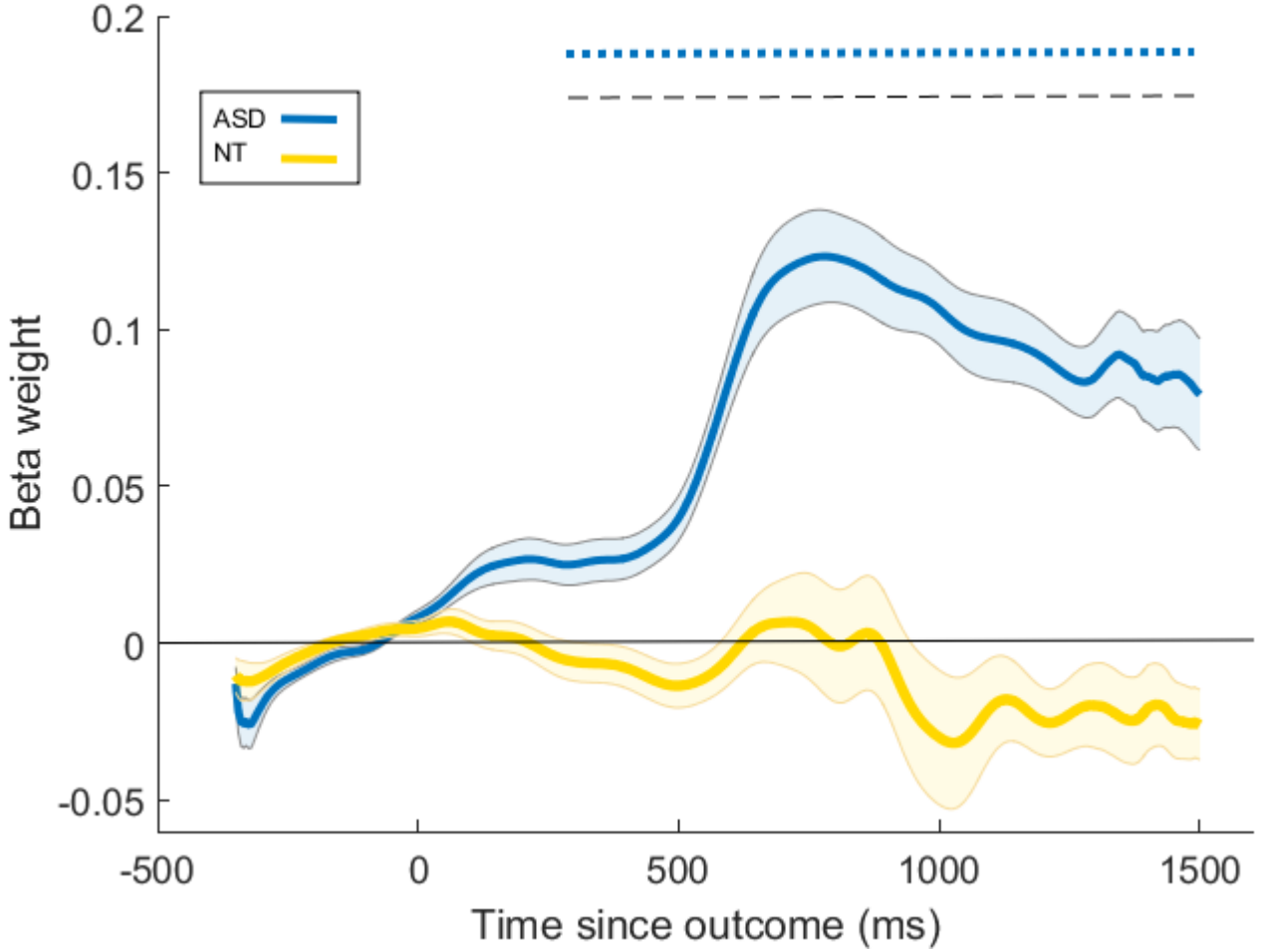






HGF: empirical evidence (Lawson et al., in revision)

Effect of precision-weighted volatility prediction error ε_3 on pupil diameter:

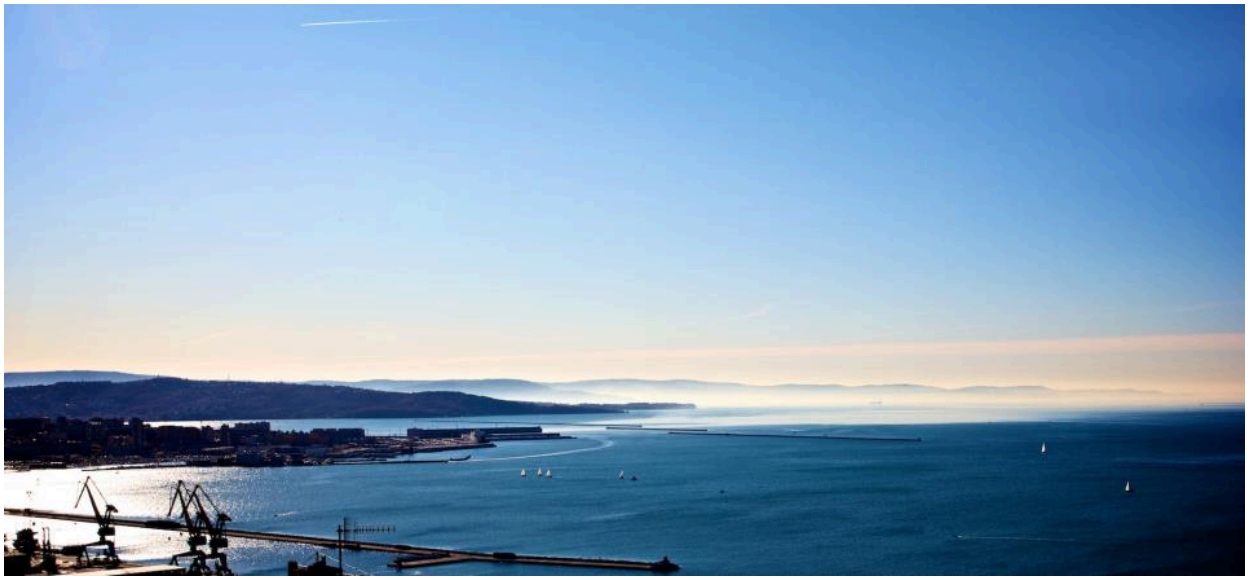


How to estimate and compare models: the HGF Toolbox

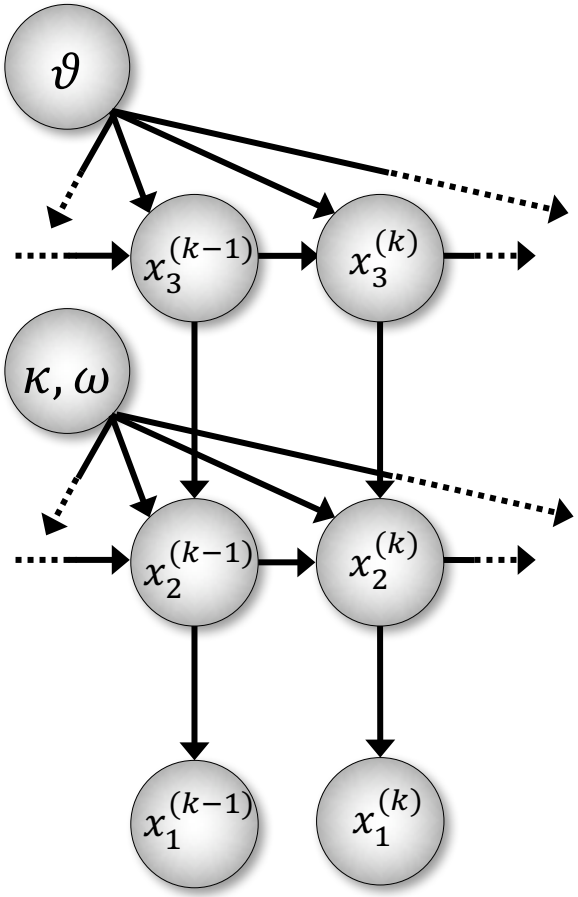
- Available at **https://www.tnu.ethz.ch/tapas**
- Start with README, manual, and interactive demo
- Modular, extensible
- Matlab-based

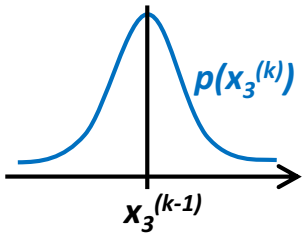
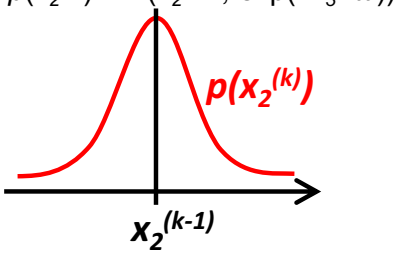
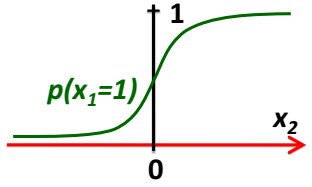
Thanks

Rick Adams
Archie de Berker
Sven Bestmann
Kay Brodersen
Jean Daunizeau
Andreea Diaconescu
Ray Dolan
Karl Friston
Sandra Iglesias
Lars Kasper
Rebecca Lawson
Ekaterina Lomakina
Berk Mirza
Read Montague
Tobias Nolte
Saee Paliwal
Robb Rutledge
Klaas Enno Stephan
Philipp Schwartenbeck
Simone Vossel
Lilian Weber



Application to binary data



State of the world	Model	
Log-volatility x_3 of tendency	Gaussian random walk with constant step size ϑ	$p(x_3^{(k)}) \sim N(x_3^{(k-1)}, \vartheta)$ 
Tendency x_2 towards category "1"	Gaussian random walk with step size $\exp(\kappa x_3+\omega)$	$p(x_2^{(k)}) \sim N(x_2^{(k-1)}, \exp(\kappa x_3+\omega))$ 
Stimulus category x_1 ("0" or "1")	Sigmoid trans-formation of x_2	$p(x_1=1) = s(x_2)$ $p(x_1=0) = 1-s(x_2)$ 

Mathys et al. (2011). *Front. Hum. Neurosci.*, 5:39.

Update equation for binary observations

- $x_1 \in \{0,1\}$ is observed by the agent. Each observation leads to an update in the belief on x_2, x_3, …, and so on up the hierarchy.
- The updates for x_2 can be derived in the same manner as above.

$$I\left(x_2^{(k)}\right) = \ln s\left(x_2^{(k)}\right) + x_2^{(k)}\left(x_1^{(k)} - 1\right) - \frac{1}{2}\hat{\pi}_2^{(k)}\left(x_2^{(k)} - \mu_2^{(k-1)}\right)^2$$

$$\mu_2^{(k)} = \mu_2^{(k-1)} + \sigma_2^{(k)}\delta_1^{(k)}$$

- At first, this simply looks like an uncertainty-weighted update. However, when we unpack σ_2 and do a Taylor expansion in powers of $\hat{\pi}_1$, we see that it is again proportional to the precision of the prediction on the level below:

$$\sigma_2^{(k)} = \frac{\hat{\pi}_1^{(k)}}{\hat{\pi}_2^{(k)}\hat{\pi}_1^{(k)} + 1} = \hat{\pi}_1^{(k)} - \hat{\pi}_2^{(k)}\left(\hat{\pi}_1^{(k)}\right)^2 + \left(\hat{\pi}_2^{(k)}\right)^2\left(\hat{\pi}_1^{(k)}\right)^3 + O(4)$$

- At all higher levels, the updates are as previously derived.

VAPEs and VOPEs

The updates of the belief on x_1 are driven by value prediction errors (VAPEs)

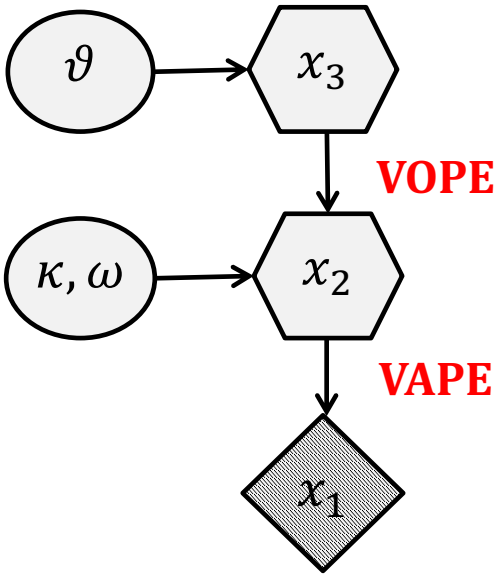
$$\mu_1^{(k)} = \mu_1^{(k-1)} + \frac{\hat{\pi}_u}{\pi_1^{(k)}}\left(u^{(k)} - \mu_1^{(k-1)}\right), \quad \textbf{VAPE}$$

while the x_2-updates are driven by volatility prediction errors (VOPEs)

$$\mu_2^{(k)} = \mu_2^{(k-1)} + \frac{1}{2}\kappa_1 v_1^{(k)} \frac{\hat{\pi}_1^{(k)}}{\pi_2^{(k)}} \delta_1^{(k)} \quad \textbf{VOPE}$$

$$\delta_1^{(k)} \stackrel{\text{def}}{=} \frac{\sigma_1^{(k)} + \left(\mu_1^{(k)} - \mu_1^{(k-1)}\right)^2}{\sigma_1^{(k-1)} + \exp\left(\kappa_1 \mu_2^{(k-1)} + \omega_1\right)} - 1$$

3-level HGF for binary observations



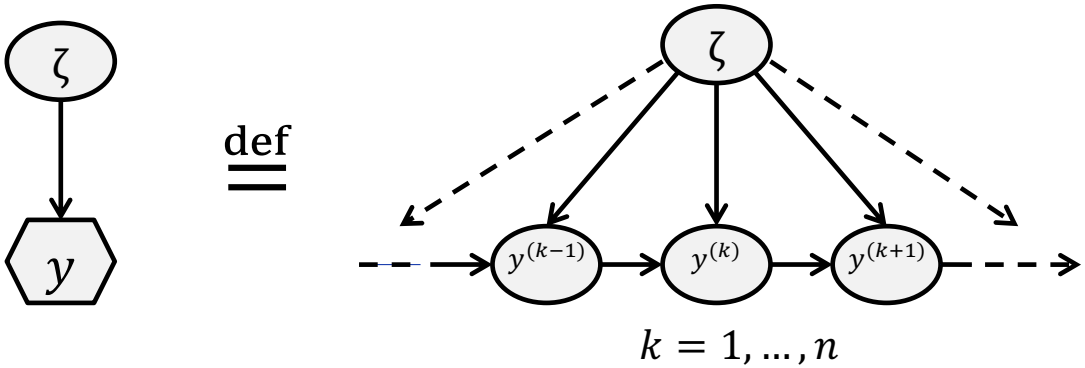
$$x_3^{(k)} \sim \mathcal{N}\left(x_3^{(k-1)}, \vartheta\right)$$

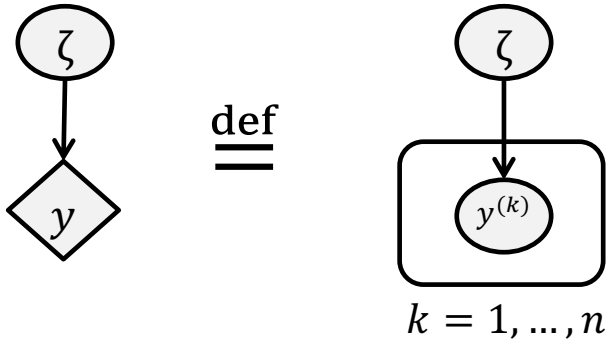
$$x_2^{(k)} \sim \mathcal{N}\left(x_2^{(k-1)}, \exp\left(\kappa x_3^{(k)} + \omega\right)\right)$$

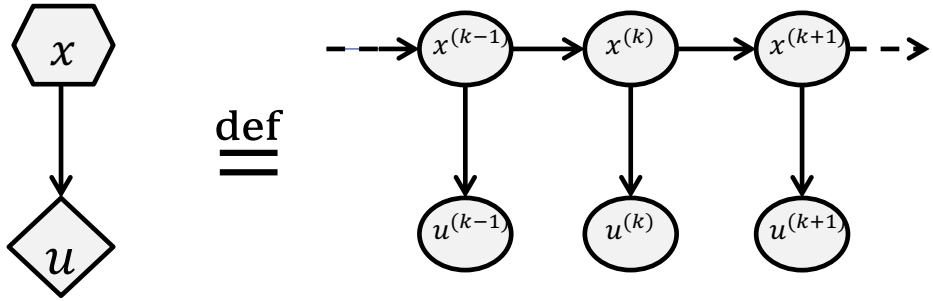
$$x_1^{(k)} \sim \text{Bern}\left(s\left(x_2^{(k)}\right)\right)$$

Mathys et al., 2011; Iglesias et al., 2013; Vossel et al., 2014a; Hauser et al., 2014; Diaconescu et al., 2014; Vossel et al., 2014b; ...

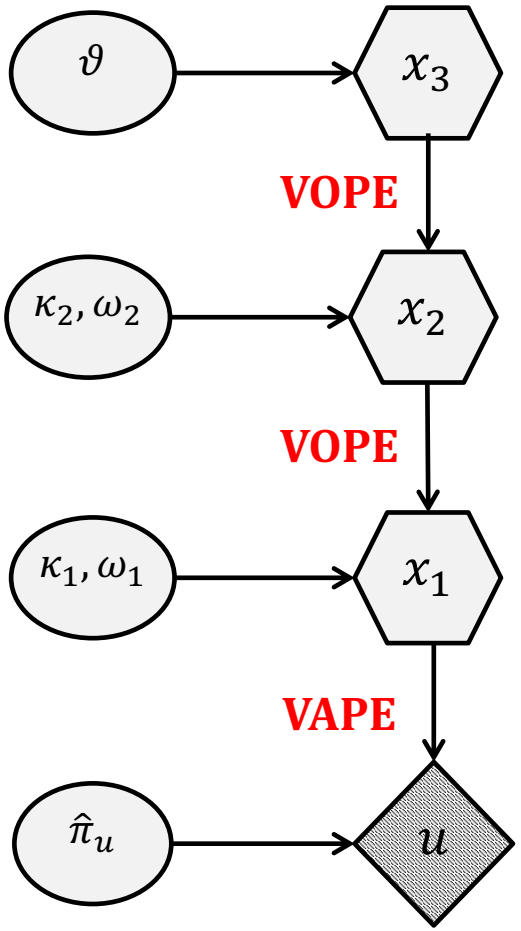
Notation


ζ
y
def
ζ
$y^{(k-1)}$
$y^{(k)}$
$y^{(k+1)}$
$k = 1, \dots, n$


ζ
y
def
ζ
$y^{(k)}$
$k = 1, \dots, n$


x
u
def
$x^{(k-1)}$
$x^{(k)}$
$x^{(k+1)}$
$u^{(k-1)}$
$u^{(k)}$
$u^{(k+1)}$
$k = 1, \dots, n$

3-level HGF for continuous observations



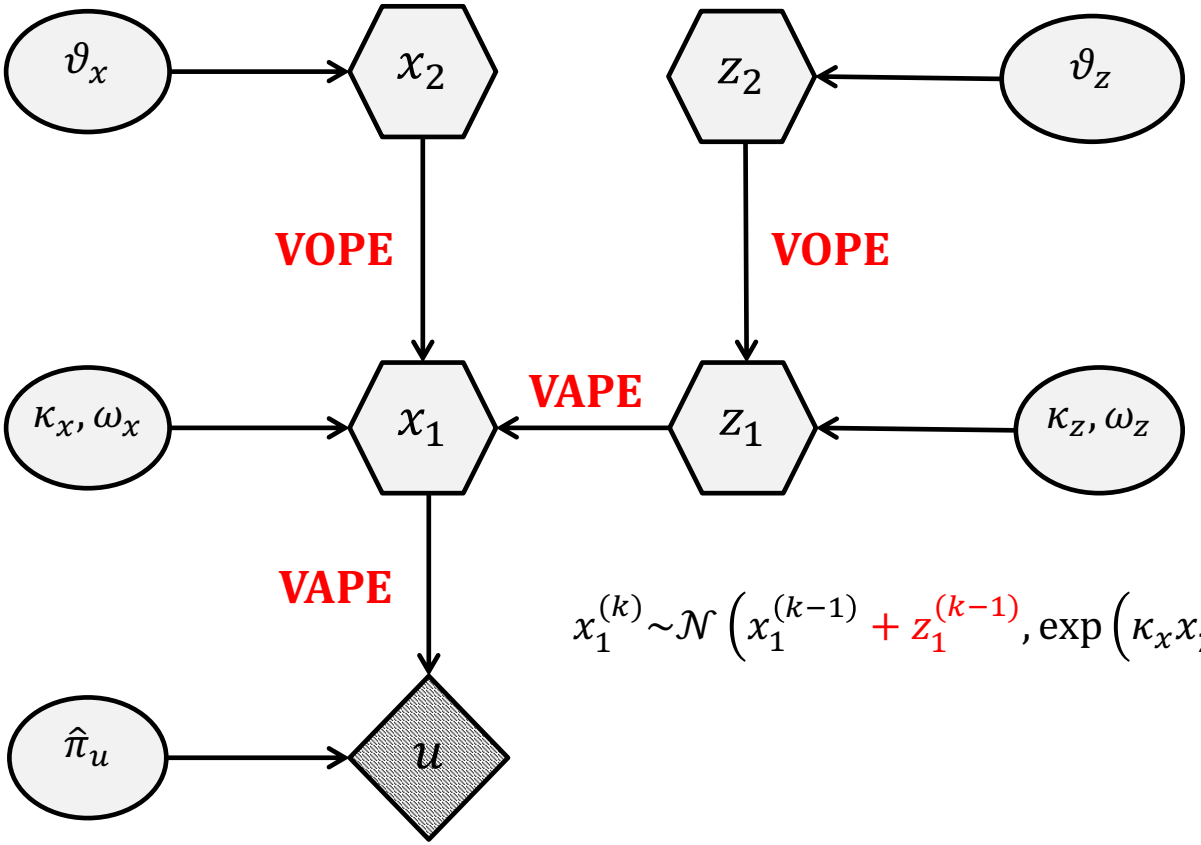
$$x_3^{(k)} \sim \mathcal{N}\left(x_3^{(k-1)}, \vartheta\right)$$

$$x_2^{(k)} \sim \mathcal{N}\left(x_2^{(k-1)}, \exp\left(\kappa_2 x_3^{(k)} + \omega_2\right)\right)$$

$$x_1^{(k)} \sim \mathcal{N}\left(x_1^{(k-1)}, \exp\left(\kappa_1 x_2^{(k)} + \omega_1\right)\right)$$

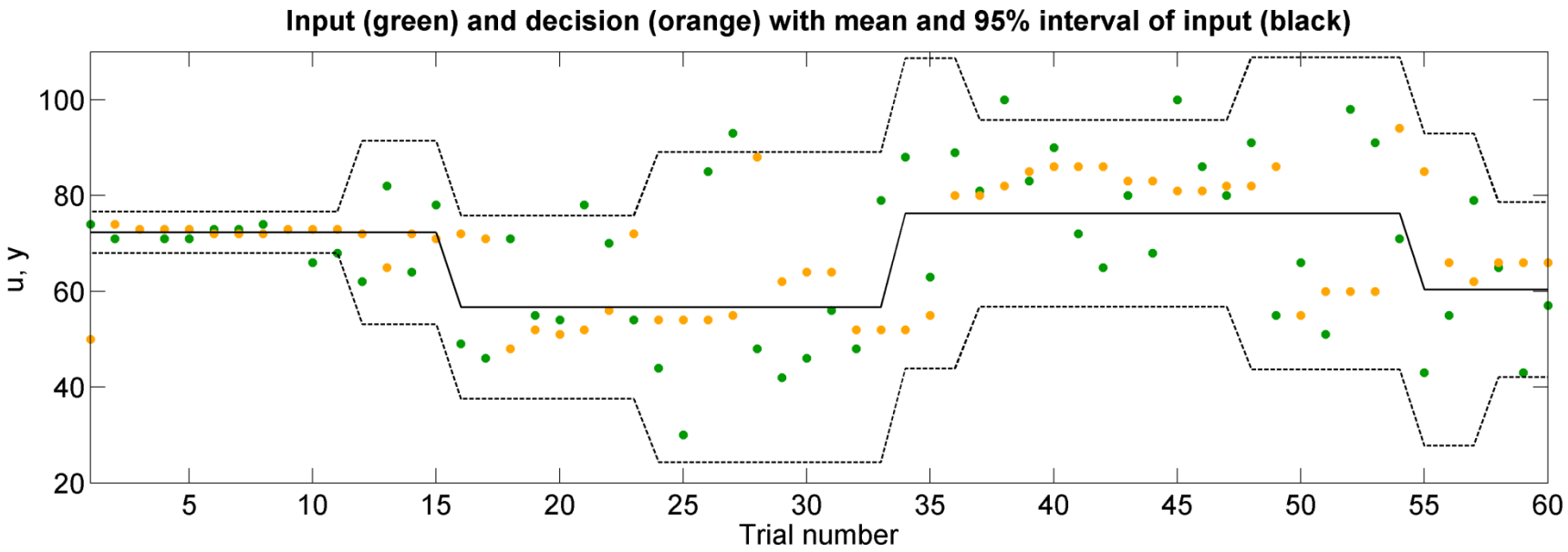
$$u^{(k)} \sim \mathcal{N}\left(x_1^{(k)}, \hat{\pi}_u^{-1}\right)$$

Variable drift



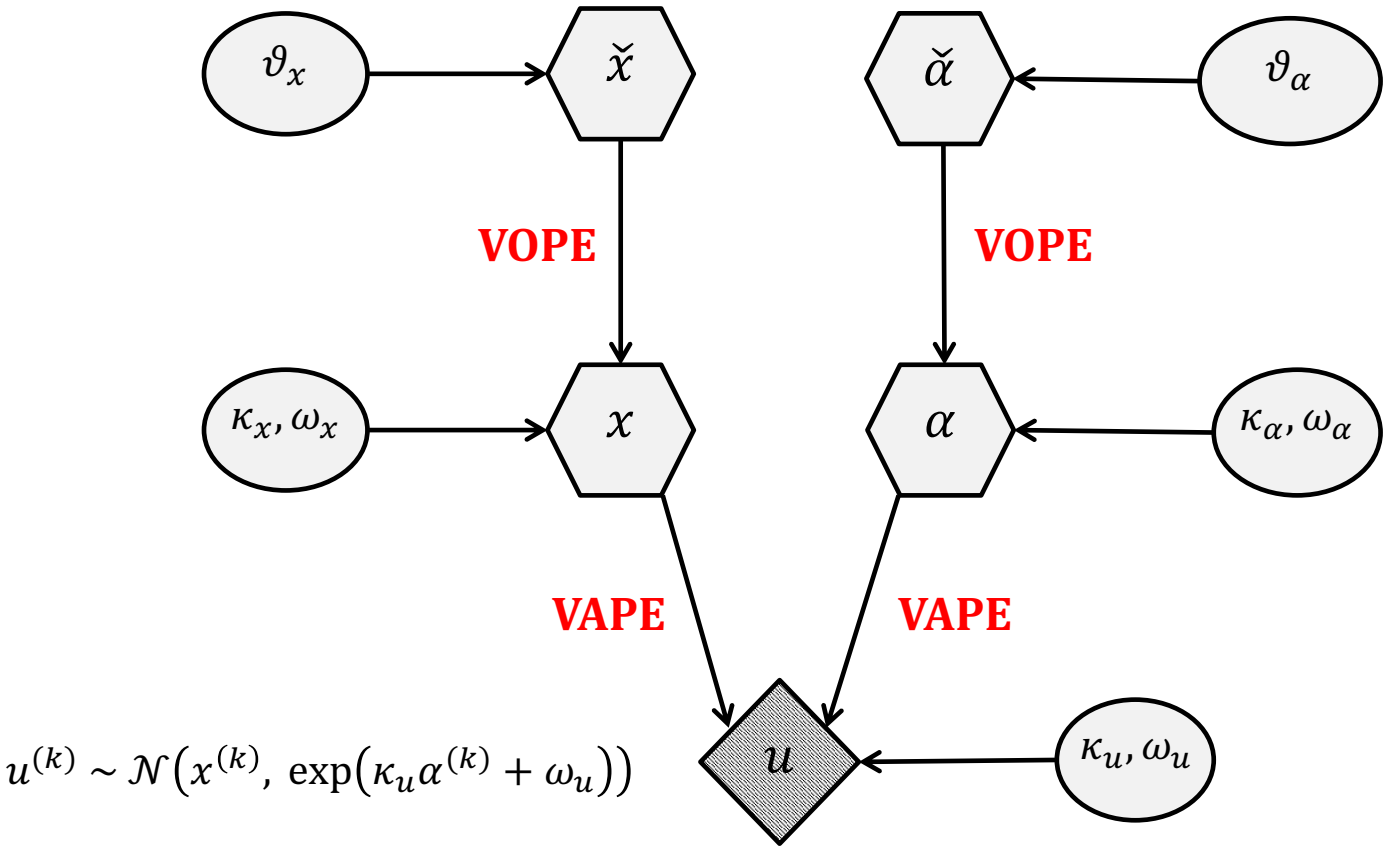
$$x_1^{(k)} \sim \mathcal{N}\left(x_1^{(k-1)} + z_1^{(k-1)}, \exp\left(\kappa_x x_2^{(k)} + \omega_x\right)\right)$$

Jumping Gaussian estimation task

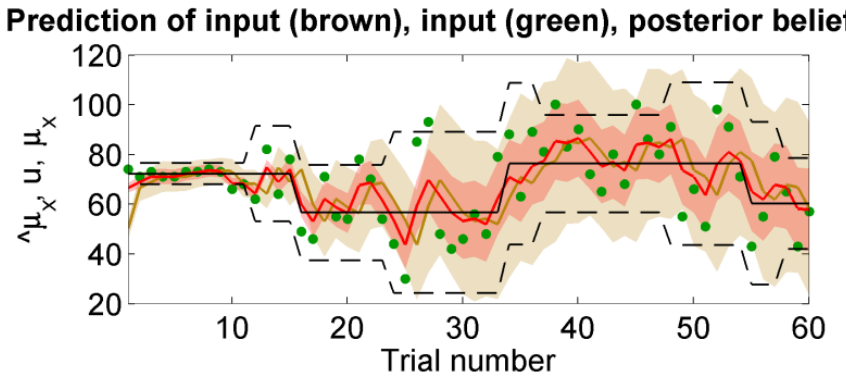


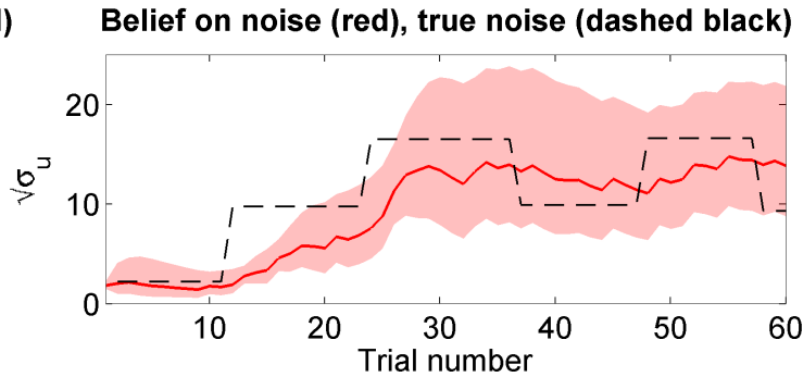
Data from Chaohui Guo

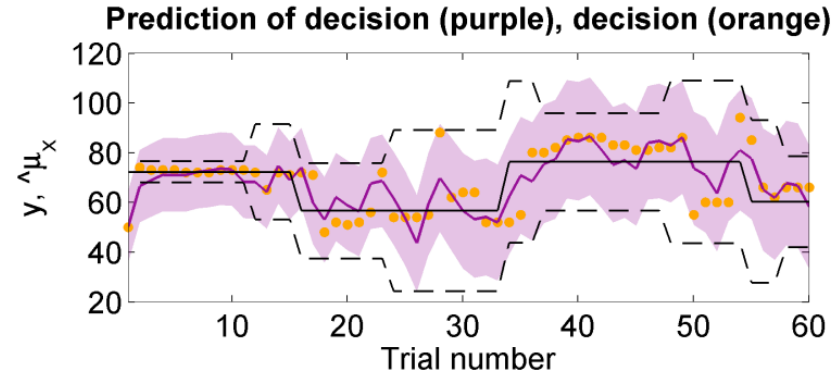
Independent mean and variance model

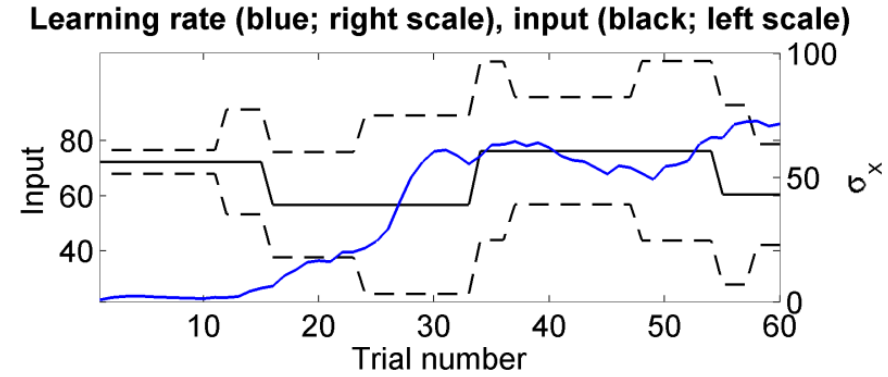


Jumping Gaussian estimation task

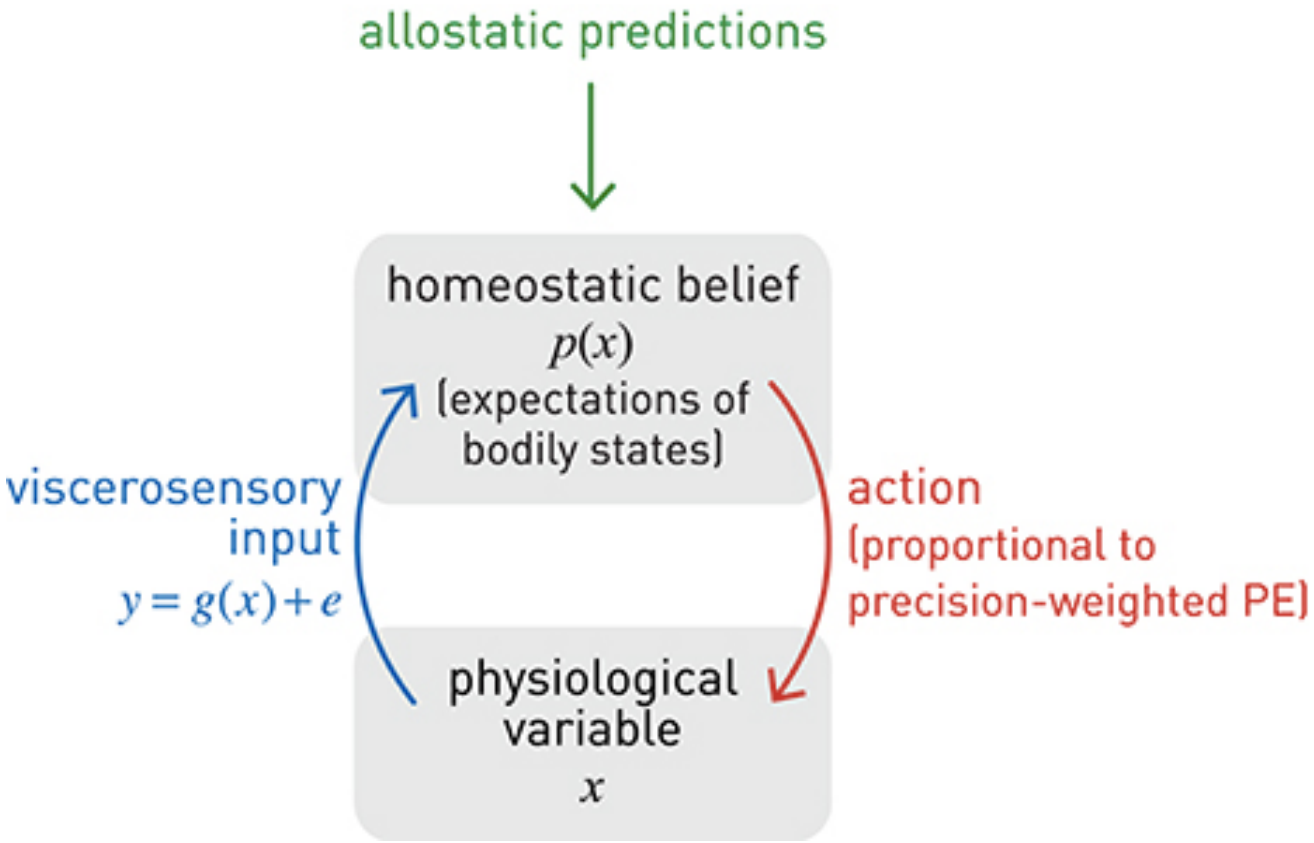








Action as active inference



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Action as active inference

Model:

$$p(x) = N\left(x; \mu_t, \pi_t^{-1}\right)$$

$$p(y \mid x) = N\left(y; g(x), \pi_{data}^{-1}\right)$$

Inference (i.e., belief update):

$$\mu_{t+1} = \mu_t + \frac{\pi_{data}}{\pi_t + \pi_{data}}\left(y_t - g(\mu_t)\right)$$

$$\pi_{t+1} = \pi_t + \pi_{data}$$

Stephan et al. (2016), Front. Hum. Neurosci., 10:550

Action as active inference

Delta priors on mean and precision of state:

$$
\begin{aligned}
p\left(y|m_H\right) &= \int p\left(y|\mu_t,\pi_t\right) p\left(\mu_t\right) p\left(\pi_t\right) d\mu_t\, d\pi_t \\
&= \int N\left(y; g\left(\mu_t\right), \pi_t^{-1}\right) \delta\left(\mu_t - \mu_{prior}\right) \\
&\quad \delta\left(\pi_t - \pi_{prior}\right) d\mu_t\, d\pi_t \\
&= N\left(y; g\left(\mu_{prior}\right), \pi_{prior}^{-1}\right)
\end{aligned}
$$

Negative of log-evidence *L* is Shannon surprise *S*:

$$
\begin{aligned}
L &= \ln p\left(y|m_H\right) \\
&= \frac{1}{2}\left(\ln \pi_{prior} - \pi_{prior}\left(y - g\left(\mu_{prior}\right)\right)^2\right) + c \\
&= \frac{1}{2}\left(\ln \pi_{prior} - \pi_{prior}\left(PE\left(y\right)\right)^2\right) + c
\end{aligned}
\qquad
S\left(y \mid m_H\right) = -L
$$

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Action as active inference

Definition of action:

$$
\begin{aligned}
a(t) &= \frac{\partial L}{\partial x} \\
&= \frac{\partial\left[-\frac{1}{2}\pi_{prior}\left(y(x,t)-g\left(\mu_{prior}\right)\right)^2\right]}{\partial x} \\
&= \frac{\partial\left[-\frac{1}{2}\pi_{prior}PE(y)^2\right]}{\partial x} \\
&= -\frac{\pi_{prior}}{2}\frac{\partial\left[PE(y)^2\right]}{\partial y}\frac{\partial y}{\partial x} \\
&= -\pi_{prior}PE(y)\frac{\partial g}{\partial x}
\end{aligned}
$$

precision-weighted prediction error

Action induces gradient descent on surprise *S*:

$$\frac{dx}{dt} = \lambda^{-1} f(a(t)) \qquad \frac{dx}{dt} = -\lambda^{-1} f\left(\frac{\partial S}{\partial x}\right)$$

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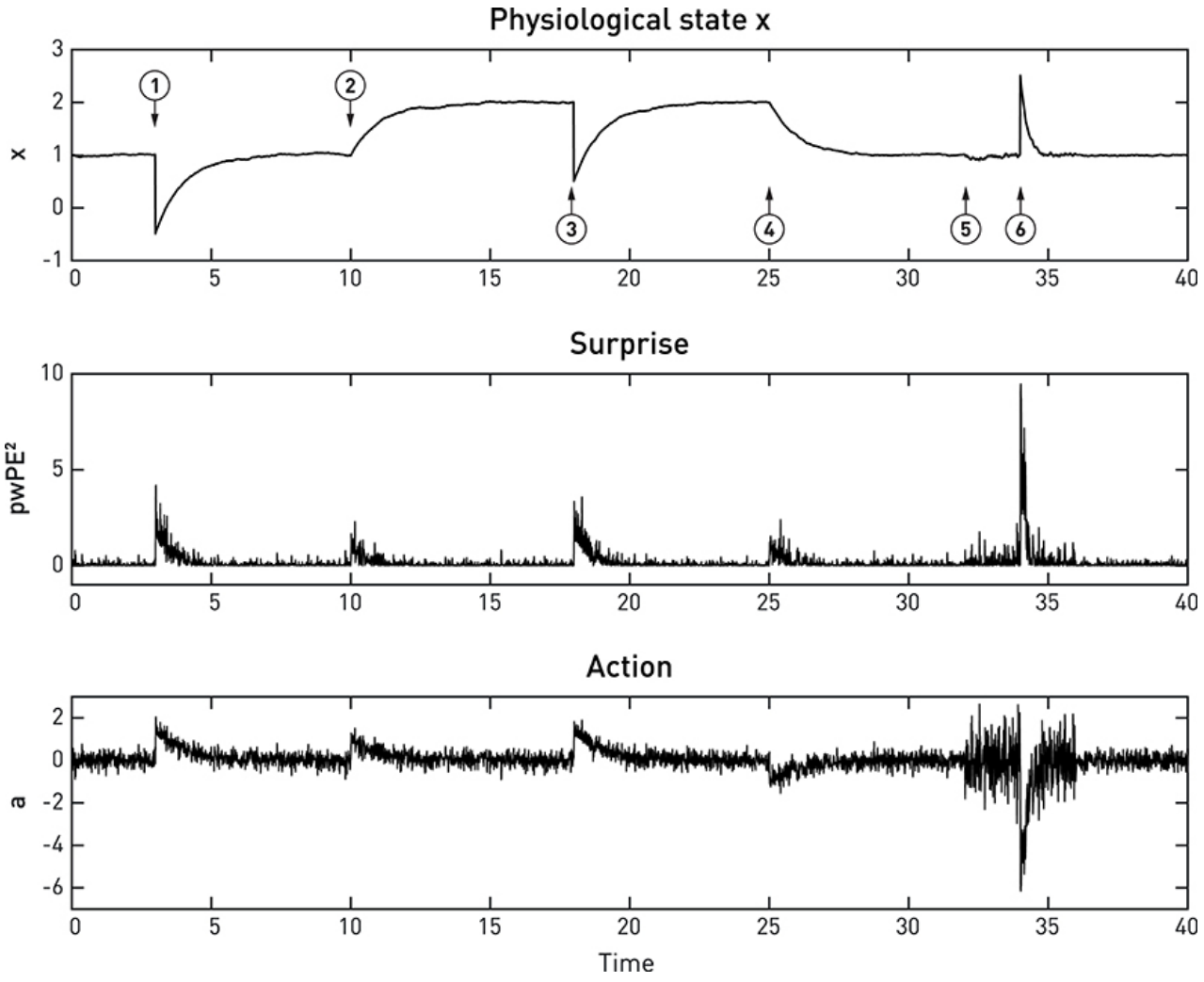
Action as active inference



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